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Abstract	We show that a class of nonlinear Fourier transforms called FBI (Fourier-Bros-Iagolintzer) transforms introduced in [6] can be used to characterize local and microlocal Gevrey regularity.	

Characterization of Gevrey Regularity by a Class of FBI Transforms

S. Berhanu and Abraham Hailu

1 Introduction

The FBI transform is a nonlinear Fourier transform introduced by J. Bros and D. Iagolintzer in order to characterize the local and microlocal analyticity of functions (or distributions) in terms of appropriate decays in the spirit of the Paley-Wiener theorem. This paper characterizes local and microlocal Gevrey regularity in terms of appropriate decays of a more general class of FBI transforms that were introduced in [6]. The classical and more commonly used FBI transform has the form

$$\mathscr{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^2} u(x')\,dx', \quad x,\xi \in \mathbb{R}^m \tag{1}$$

where u is a continuous function of compact support in $\mathbb{R}^m$ or a distribution of compact support in which case the integral is understood in the duality sense. This transform characterizes microlocal analyticity (see [14]) and microlocal smoothness (see [8]) and has been used in numerous works to study the regularity of solutions of linear and nonlinear partial differential equations.

Among the many works where the transform (1.1) has been used, we mention [2–5, 7–12] and [14]. In [14] (see also [8] and [15]) more general FBI transforms

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than (1.1) were considered where the phase function behaved much like the quadratic phase $i\xi \cdot (x - x') - |\xi||x - x'|^2$ in that the real part of the Hessian was required to be negative definite.

In the work [6] the authors introduced a more general class of FBI transforms where the real part of the Hessian of the phase function may degenerate at the point of interest. It was shown that these more general transforms characterize local and microlocal smoothness and real analyticity. Simple examples of the transforms that were introduced include, for each $k = 2, 3, \dots,$

$$\mathscr{F}_k u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^{2k}} u(x')\, dx', \quad x, \xi \in \mathbb{R}^m.$$

Observe that for $k > 1$, these transforms have a degenerate Hessian at the origin. In [6] $\mathscr{F}_2 u$ was used to establish the microlocal hypoellipticity of certain systems of complex vector fields in a situation where the standard transform $\mathscr{F}u$ didn't seem to help.

In section 2 we discuss the local and microlocal characterization of Gevrey functions as boundary values of almost analytic functions F with the property that $\overline{\partial} F$ decays exponentially. In section 3 we present a characterization of the Gevrey wave front set in terms of appropriate decays of a class of FBI transforms introduced in [6]. This result generalizes a result of M. Christ ([7]) who proved a similar characterization using the classical transform given by (1.1).

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2 Gevrey Functions and Some Preliminaries

Definition 1 Let $s \geq 1$. Let $f(x) \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^m$ open. The function f is a Gevrey function of order s on Ω if for any $K \subset\subset \Omega$ there is a constant $C_K > 0$ such that

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1} (\alpha!)^s, \ \forall\, x \in K, \ \forall\, \alpha.$$

We denote the class of Gevrey functions of order s on Ω by $G^s(\Omega)$. If $s = 1$, then $G^1(\Omega) = C^\omega(\Omega)$ is the space of real analytic functions on Ω.

Definition 2 Let $\Omega \subset \mathbb{R}^m$ be open, and $u \in \mathscr{D}'(\Omega)$, $s > 1$. Let $x_0 \in \Omega$. We say $(x_0, \xi^0) \notin WF_s(u)$ (Gevrey wave front set of u) if there is $\varphi \in G^s \cap C_0^\infty$ (Gevrey function of compact support), $\varphi \equiv 1$ near x_0, a conic neighborhood Γ of ξ^0 and constants $c_1, c_2 > 0$ such that

$$|\widehat{\varphi u}(\xi)| \leq c_1 \exp\left(-c_2 |\xi|^{\frac{1}{s}}\right), \ \ \forall \xi \in \Gamma.$$

Equivalently,

$$|\widehat{\varphi u}(\xi)| \leq c_1^{N+1}(N!)|\xi|^{\frac{-N}{s}}, \ \ \forall \xi \in \Gamma, \forall N = 1, 2, \ldots .$$

Here $\widehat{\varphi u}(\xi)$ denotes the Fourier transform of φu.

It is well known that $u \in G^s(\Omega)$ if and only if $WF_s(u) = \emptyset$ over Ω (see [13]).

Theorem 1 *Let $\Omega \subset \mathbb{R}^m$ be open. $f \in G^s(\Omega)$ if and only if for each $K \subset\subset \Omega$ relatively compact and open, there is $F(x, y) \in C^1(K \times \mathbb{R}^m)$ such that*

1. $F(x, 0) = f(x)$ on K and
2.

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x, y)\right| \leq c_1 \exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1, 2, \ldots, m$$

on $K \times B_\delta$ for some constants $c_1, c_2, \delta > 0$ where $B_\delta = \{y \in \mathbb{R}^m : |y| < \delta\}$ and $z_j = x_j + iy_j$.

In the proof of Theorem (1) we will use the following remark.

Remark 1 It is easy to see that condition (2) in Theorem (1) holds if and only if for some $c > 0$

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x, y)\right| \leq c^{N+1} N! |y|^{\frac{N}{s-1}}, \forall N = 0, 1, 2, \ldots . \tag{2}$$

Proof Suppose $f(x) \in G^s(\Omega)$ and $K \subset\subset \Omega$ relatively compact and open. Let $\{a_{|\alpha|}\}_{|\alpha| \in \mathbb{N}}$ be defined by

$$a_{|\alpha|} = \frac{1}{C|\alpha|^{s-1}}, \ \ a_0 = 1$$

for some C to be chosen later. Set

$$F(x, y) = \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha f(x) y^\alpha \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \tag{3}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, $\chi(x) \equiv 0$ when $|x| \geq 1$, $0 \leq \chi \leq 1$.

We will first show that F is C^1. Since $f(x) \in G^s$, there is $C_K > 0$ such that

$$\left|\partial_x^\alpha f(x)\right| \leq C_K^{|\alpha|+1}(\alpha!)^s, \ \ \forall\, x \in K, \ \forall\, \alpha. \tag{4}$$

For $x \in K$, since χ is supported in $[-1, 1]$, 68

$$\left| \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha f(x) y^\alpha \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \right| \le C_K^{|\alpha|+1} (\alpha!)^{s-1} \frac{1}{C^{|\alpha|} |\alpha|^{|\alpha|(s-1)}}$$
$$\le C \left(\frac{C_K}{C}\right)^{|\alpha|+1} \tag{5}$$

For each α, let $g_\alpha(x, y) = \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha f(x) y^\alpha \chi\left(\frac{|y|}{a_{|\alpha|}}\right)$. 69

$$|\partial_{x_j} g_\alpha(x, y)| \le \frac{C_K^{|\alpha|+2}}{\alpha!} ((\alpha + e_j)!)^s \frac{1}{C^{|\alpha|} |\alpha|^{|\alpha|(s-1)}}$$
$$\le \frac{C_K^{|\alpha|+2}}{\alpha!} 2^{s|\alpha|} (\alpha!)^s \frac{1}{C^{|\alpha|} |\alpha|^{|\alpha|(s-1)}}$$
$$\le C^2 \left(2^s \frac{C_K}{C}\right)^{|\alpha|+2} \tag{6}$$

where we used the fact that $(\alpha + e_j)! \le 2^{|\alpha|} \alpha!$. Next we consider 70

$$\partial_{y_j} g_\alpha(x, y) = \frac{\alpha_j i^{|\alpha|}}{\alpha!} y^{\alpha - e_j} (\partial_x^\alpha f)(x) \chi\left(\frac{|y|}{a_{|\alpha|}}\right) + \frac{i^{|\alpha|}}{\alpha!} y^\alpha (\partial_x^\alpha f)(x) \chi'\left(\frac{|y|}{a_{|\alpha|}}\right) \frac{y_j}{a_{|\alpha|} |y|}$$
$$= A_\alpha(x, y) + B_\alpha(x, y). \tag{7}$$

Here if $\alpha_j = 0$, we set $A_\alpha(x, y) = 0$. We have: 71

$$|A_\alpha(x, y)| \le C^2 \left(\frac{C_K}{C}\right)^{|\alpha|+1} |\alpha|^s$$

and 72

$$|B_\alpha(x, y)| \le C^2 C' \left(\frac{C_K}{C}\right)^{|\alpha|+1} |\alpha|^{s-1}, \;\; C' = \sup \chi'.$$

It follows that 73

$$|\partial_{y_j} g_\alpha(x, y)| \le C^2 (1 + C') \left(\frac{C_K}{C}\right)^{|\alpha|+1} |\alpha|^s$$

We now choose $C = 2^s C_K$. From the preceding estimates, we conclude that F is C^1. 74

We next compute $\frac{\partial F}{\partial \bar{z}_j}(x, y)$ for each $j = 1, \ldots, m$. Fix $j = 1, \ldots, m$. Then

$$\begin{aligned}
\frac{\partial F}{\partial \bar{z}_j}(x, y) &= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\partial_x^{\alpha} f(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right) \\
&\quad + \frac{i}{2}\frac{\partial}{\partial y_j}\left(\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\partial_x^{\alpha} f(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right) \\
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{\{\alpha:\alpha_j \geq 1\}} \frac{\alpha_j i^{|\alpha|}}{\alpha!} y^{\alpha-e_j} (\partial_x^{\alpha} f)(x) \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha} (\partial_x^{\alpha} f)(x) \chi'\left(\frac{|y|}{a_{|\alpha|}}\right) \frac{y_j}{a_{|\alpha|}|y|}
\end{aligned}$$

where $e_j = (0, \ldots, 0, \underbrace{1}_{\text{jth place}}, 0, \ldots,) \in \mathbb{N}_0^m$.

Let $\beta = \alpha - e_j$. Then $|\beta| = |\alpha| - |e_j| \geq 0$ in the second sum and so

$$\begin{aligned}
\frac{\partial F}{\partial \bar{z}_j}(x, y) &= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{|\beta| \geq 0} \frac{(\beta_j + 1) i^{|\beta+e_j|}}{(\beta + e_j)!} y^{\beta} (\partial_x^{\beta+e_j} f)(x) \chi\left(\frac{|y|}{a_{|\beta+e_j|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha} (\partial_x^{\alpha} f)(x) \chi'\left(\frac{|y|}{a_{|\alpha|}}\right) \frac{y_j}{a_{|\alpha|}|y|} \\
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{1}{2}\sum_{\beta} \frac{1}{\beta!} i^{|\beta|+1+1} y^{\beta} (\partial_x^{\beta+e_j} f)(x) \chi\left(\frac{|y|}{a_{|\beta+e_j|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha} (\partial_x^{\alpha} f)(x) \chi'\left(\frac{|y|}{a_{|\alpha|}}\right) \frac{y_j}{a_{|\alpha|}|y|} \\
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)
\end{aligned}$$

$$+\frac{i}{2}\sum_{\alpha}\frac{i^{|\alpha|}}{\alpha!}y^{\alpha}(\partial_x^{\alpha}f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|}$$

$$=\Sigma_1(x,y)+\Sigma_2(x,y) \tag{8}$$

We observe that

$$\Sigma_1(x,y)\neq 0\Rightarrow \frac{1}{2}\leq\frac{|y|}{a_{|\alpha|+1}}\ \text{and}\ \frac{|y|}{a_{|\alpha|}}\leq 1$$

and so

$$\frac{a_{|\alpha|+1}}{2}\leq |y|\leq a_{|\alpha|}.$$

Then by the definition of the $a_{|\alpha|}$ we get

$$\Sigma_1(x,y)\neq 0\Rightarrow\frac{1}{2C|(\alpha|+1)^{s-1}}\leq|y|\leq\frac{1}{C|\alpha|^{s-1}}. \tag{9}$$

Each term in $\Sigma_1(x,y), x\in K$ satisfies

$$\begin{aligned}
&\left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j}f\right)(x)y^{\alpha}\left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right)-\chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right|\\
&\leq\frac{2|y|^{|\alpha|}}{\alpha!}C_K^{|\alpha+e_j|+1}((\alpha+e_j)!)^s\\
&\leq\frac{2}{\alpha!}\left(\frac{1}{C|\alpha|^{s-1}}\right)^{|\alpha|}C_K^{|\alpha+e_j|+1}((\alpha+e_j)!)^s,\quad\text{by (9)}\\
&\leq\frac{2}{\alpha!}\left(\frac{1}{C|\alpha|^{s-1}}\right)^{|\alpha|}C_K^{|\alpha+e_j|+1}(\alpha!)^s(e_j!)^s2^{s(|\alpha|+1)},\quad\text{using }(\beta+\delta)!\leq\beta!\delta!2^{|\beta|+|\delta|}\\
&=C_K'\left(\frac{2^sC_K}{C}\right)^{|\alpha|+1}\left(\frac{\alpha!}{|\alpha|^{|\alpha|}}\right)^{s-1},\quad C_K'=2CC_K\\
&\leq C_K'\left(\frac{2^sC_K}{C}\right)^{|\alpha|+1}\left(\frac{|\alpha|!}{|\alpha|^{|\alpha|}}\right)^{s-1}\\
&\leq C_K'\left(\frac{2^sC_K}{C}\right)^{|\alpha|+1}\left(\frac{\sqrt{2\pi|\alpha|}}{e^{|\alpha|-1}}\right)^{s-1},\quad\text{(by Stirling's formula)}
\end{aligned} \tag{10}$$

From inequality (9) we have

$$\frac{1}{(2C)^{\frac{1}{s-1}}}\frac{1}{|y|^{\frac{1}{s-1}}}\leq|\alpha|+1\Rightarrow\frac{1}{|y|^{\frac{1}{s-1}}}\left(\frac{1}{(2C)^{\frac{1}{s-1}}}-|y|^{\frac{1}{s-1}}\right)\leq|\alpha|.$$

Thus if $|y|$ is small, say $|y|^{\frac{1}{s-1}} < \frac{1}{2(2C)^{\frac{1}{s-1}}}$ and $\Sigma_1(x,y) \neq 0$, then we get 83

$$\frac{1}{|y|^{\frac{1}{s-1}}}\left(\frac{1}{(2C)^{\frac{1}{s-1}}} - \frac{1}{2(2C)^{\frac{1}{s-1}}}\right) \leq \frac{1}{|y|^{\frac{1}{s-1}}}\left(\frac{1}{(2C)^{\frac{1}{s-1}}} - |y|^{\frac{1}{s-1}}\right) \leq |\alpha|.$$

Hence, 84

$$\frac{A_s}{|y|^{\frac{1}{s-1}}} \leq |\alpha|, \; A_s = \frac{1}{2(2C)^{\frac{1}{s-1}}}.$$

Thus, 85

$$\frac{1}{|\alpha|^{N+1}} \leq \frac{|y|^{\frac{N+1}{s-1}}}{A_s^{N+1}}, \quad N = 0, 1, 2, \ldots . \tag{11}$$

From (10) and (11) we get 86

$$\begin{aligned}
&\left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\leq C'_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{\sqrt{2\pi|\alpha|}}{e^{|\alpha|-1}}\right)^{s-1} \\
&= C''_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} e^{-|\alpha|(s-1)}, \quad C''_K = C'_K e^{s-1} \sqrt{2\pi}^{s-1} > 0 \\
&\leq C''_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} \frac{(N+1)!}{(s-1)^{N+1}} \frac{1}{|\alpha|^{(N+1)}}, \quad N = 0, 1, 2, \ldots \\
&\leq \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} |y|^{\frac{N+1}{s-1}}.
\end{aligned} \tag{12}$$

Thus using (12), we get 87

$$\begin{aligned}
&\left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\leq \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} e^{|\alpha|\left(\frac{s-1}{2}\right)} |y|^{\frac{N+1}{s-1}} \\
&\leq D_1^{N+1} (N+1)! \left(\frac{2^s C_K e^{\frac{s-1}{2}}}{C}\right)^{|\alpha|+1} |y|^{\frac{N+1}{s-1}}, D_1 = \frac{C''_K + 1}{(s-1)A_s}
\end{aligned}$$

$$\le D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}}$$
$$\left(\text{we may assume } C \text{ was chosen so that } \frac{2^s C_K e^{\frac{s-1}{2}}}{C} \le 1\right)$$

Thus

$$\left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \le D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}},\ N = 0, 1, 2, \ldots \tag{13}$$

From equation (9), when $\Sigma_1(x, y) \neq 0$, we have

$$|\alpha| \le \frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}.$$

Therefore, using this and inequality (13), we have

$$\begin{aligned}
|\Sigma_1(x, y)| &\le \sum_{|\alpha| \le \frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}} D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}},\quad N = 0, 1, 2, \ldots \\
&= D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}} \sum_{|\alpha| \le \frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}} 1 \\
&\le D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}} \frac{1}{C^{\frac{m}{s-1}}|y|^{\frac{m}{s-1}}} \\
&\le D_3^{k+1} k! |y|^{\frac{k}{s-1}},\quad k = 0, 1, 2, \ldots\quad D_3 \text{ independent of } k.
\end{aligned} \tag{14}$$

Consider $\Sigma_2(x, y)$: Since $\chi \equiv 0$ outside $(-1, 1)$ and $\chi \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we see that $\chi' \equiv 0$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and outside $(-1, 1)$. Thus

$$\Sigma_2(x, y) = \frac{i}{2}\sum_\alpha \frac{i^{|\alpha|}}{\alpha!} y^\alpha (\partial_x^\alpha f)(x) \chi'\left(\frac{|y|}{a_{|\alpha|}}\right) \frac{y_j}{a_{|\alpha|}|y|} \neq 0,$$
$$\Rightarrow \frac{1}{2} \le \frac{|y|}{a_{|\alpha|}} \le 1 \Rightarrow \frac{a_{|\alpha|}}{2} \le |y| \le a_{|\alpha|}.$$

By the same method as we used for the estimate of $\Sigma_1(x, y)$, there is $D_4 > 0$ such that

$$|\Sigma_2(x, y)| \le D_4^{N+1} N! |y|^{\frac{N}{s-1}},\quad N = 0, 1, 2, \ldots. \tag{15}$$

Combining (14) and (15), we have for some $A > 0$

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x, y)\right| \leq A^{N+1} N! |y|^{\frac{N}{s-1}}, \quad N = 0, 1, 2, \ldots, \ \forall\, j = 1, 2, \ldots, m$$

and hence (2) in Theorem 2.1 holds. Conversely, suppose that for each $K \subset\subset \Omega$ there is $F(x, y) \in C^1(K \times \mathbb{R}^m)$ such that

1. $F(x, 0) = f(x)$ and
2. $$\left|\frac{\partial F}{\partial \bar{z}_j}(x, y)\right| \leq c^{N+1} N! |y|^{\frac{N}{s-1}}, \quad j = 1, 2, \ldots, m$$

   for some constant $c > 0$.

We wish to show that $f(x) \in G^s(\Omega)$. It is sufficient to show that $f \in G^s(B)$ for each sufficiently small ball in Ω. Let B_{2r} be a ball of radius $2r$ whose closure is in Ω, and let $F(x, y)$ be given as above on a neighborhood of the closure of $\Omega_r = B_{2r} \times B_r$. We may assume that $F(x, y) \equiv 0$ for $|y| \geq r$.

Set

$$\omega(z) = dz_1 \wedge \ldots \wedge dz_m.$$

For $n \geq 1$, let σ_n denotes the area of the unit sphere S^{n-1} in $\mathbb{R}^n$. We will identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$. For $k = 1, \ldots, m$, let

$$\omega_k(\bar{z}) = (-1)^{k-1} d\bar{z}_1 \wedge \ldots d\bar{z}_{k-1} \wedge \widehat{d\bar{z}_k} \wedge d\bar{z}_{k+1} \wedge \ldots d\bar{z}_m$$

where $d\bar{z}_k$ is removed. For each $x \in B_r$, from the higher dimensional version of the inhomogeneous Cauchy Integral Formula, we have

$$\begin{aligned} f(x) = F(x, 0) &= \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\partial\Omega_r} F(w) \sum_{k=1}^{m} (\overline{w_k} - x_k)|w - x|^{-2m} \omega_k(\overline{w}) \wedge \omega(w) \\ &\quad - \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^{m} \frac{\partial F}{\partial \overline{w_k}}(w)(\overline{w_k} - x_k)|w - x|^{-2m} \omega(\bar{w}) \wedge \omega(w) \\ &= g(x) + h(x) \end{aligned} \tag{16}$$

Clearly, $g(x)$ is real analytic on B_r. If we show $h \in G^s(B_r)$, we will be done. For each $\alpha = (\alpha_1, \ldots, \alpha_m)$, we have

$$\partial^\alpha h(x) = -\frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^{m} \frac{\partial F}{\partial \overline{w_k}}(w) \partial_x^\alpha \left((\overline{w_k} - x_k)|w - x|^{-2m}\right) \omega(\bar{w}) \wedge \omega(w) \tag{17}$$

For $x \neq w$,

$$\partial_x^\alpha \left((\overline{w_k} - x_k)|w - x|^{-2m} \right) = \sum_{\beta \le \alpha} \frac{\alpha!}{(\alpha - \beta)!\beta!} \partial_x^\beta (\bar{w}_k - x_k) \partial_x^{\alpha - \beta} \left(|w - x|^{-2m} \right)$$

$$= (\overline{w_k} - x_k) \partial_x^\alpha \left(|w - x|^{-2m} \right) - \frac{\alpha!}{(\alpha - e_k)!} \partial_x^{\alpha - e_k} \left(|w - x|^{-2m} \right)$$

$$= (\overline{w_k} - x_k) \partial_x^\alpha \left(|w - x|^{-2m} \right) - \alpha_k \partial_x^{\alpha - e_k} \left(|w - x|^{-2m} \right). \tag{18}$$

We have

$$\partial_x^\alpha \left(|w - x|^{-2m} \right) = \sum_{\beta \le \alpha} a_\beta (w - x)^\beta |w - x|^{-2m - |\beta| - |\alpha|}, \text{ and so}$$

$$\partial_x^{\alpha - e_k} \left(|w - x|^{-2m} \right) = \sum_{\beta \le \alpha - e_k} b_\beta (w - x)^\beta |w - x|^{-2m - |\beta| - |\alpha| + 1}. \tag{19}$$

where a_β and b_β are constants. Plugging (19) into (18) results in

$$\begin{aligned} &\left| \partial_x^\alpha \left((\overline{w_k} - x_k)|w - x|^{-2m} \right) \right| \\ &\le |w - x| \left| \partial_x^\alpha \left(|w - x|^{-2m} \right) \right| + \alpha_k \left| \partial_x^{\alpha - e_k} \left(|w - x|^{-2m} \right) \right| \\ &\le \sum_{\beta \le \alpha} |a_\beta| |w - x|^{-2m - |\alpha| + 1} + \alpha_k \sum_{\beta \le \alpha - e_k} |b_\beta| |w - x|^{-2m - |\alpha| + 1} \\ &\le C_1 (|\alpha| + 1)^m |w - x|^{-2m - |\alpha| + 1} \end{aligned} \tag{20}$$

Using the hypothesis, equation (17) and inequality (20), we have

$$\begin{aligned} |\partial^\alpha h(x)| &\le \frac{2^{1-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^m \left| \frac{\partial F}{\partial \overline{w_k}}(w) \right| \left| \partial_x^\alpha \left((\overline{w_k} - x_k)|w - x|^{-2m} \right) \right| |\omega(\bar{w}) \wedge \omega(w)| \\ &\le \frac{22^{-m}}{\sigma_{2m}} C_1 (|\alpha| + 1)^m c^{N+1} N! \int_{\Omega_r} \sum_{k=1}^m \frac{|\Im w|^{\frac{N}{s-1}}}{|w - x|^{2m + |\alpha| - 1}} |\omega(\bar{w}) \wedge \omega(w)| \\ &\le \frac{22^{-m}}{\sigma_{2m}} C_1 (|\alpha| + 1)^m c^{N+1} N! \int_{\Omega_r} \sum_{k=1}^m \frac{|\Im w|^{\frac{N}{s-1}}}{|\Im w|^{2m + |\alpha| - 1}} |\omega(\bar{w}) \wedge \omega(w)| \\ &\le C_2^{N+1} (|\alpha| + 1)^m N! \int_{\Omega_r} |\Im w|^{\frac{N}{s-1} - (2m + |\alpha| - 1)} |\omega(\bar{w}) \wedge \omega(w)| \\ &\le C_2^{N+1} (|\alpha| + 1)^m N^N \int_{\Omega_r} |\Im w|^{\frac{N}{s-1} - (2m + |\alpha| - 1)} |\omega(\bar{w}) \wedge \omega(w)| \end{aligned} \tag{21}$$

for some $C_2 > 0$. Choose N such that

$$2m + |\alpha| - 1 \le \frac{N}{s-1} \le 2m + |\alpha| + \frac{1}{s-1}.$$

Then

$$|\Im w|^{\frac{N}{s-1} - (2m+|\alpha|-1)} \le (|\Im w| + 1)^{\frac{s}{s-1}} .$$

Since $N \le s(2m + |\alpha|)$), (21) becomes

$$\begin{aligned}
|\partial^\alpha h(x)| &\le (C_2+1)^{s(2m+|\alpha|)+1}(|\alpha|+1)^m (s(2m+|\alpha|))^{s(2m+|\alpha|)} \int_{\Omega_r} (|\Im w|+1)^{\frac{s}{s-1}} |\omega(\bar{w}) \wedge dw| \\
&= C'(C_2+1)^{s(2m+|\alpha|)+1}(|\alpha|+1)^m (s(2m+|\alpha|))^{s(2m+|\alpha|)} \\
&\le A_1^{|\alpha|+1}(2m+|\alpha|)^{s(2m+|\alpha|)}, \quad \text{some } A_1 > 0 \\
&\le A_1^{|\alpha|+1} e^{s(2m+|\alpha|)}((2m+|\alpha|)!)^s, \text{ we used } N^N \le e^N N! \\
&\le A_2^{|\alpha|+1}((2m+|\alpha|)!)^s \quad \text{some } A_2 > 0 \\
&\le A_2^{|\alpha|+1} 2^{s(2m+|\alpha|)}((2m)!)^s(|\alpha|!)^s, \quad \text{we used } (j+k)! \le 2^{j+k} k! j! \\
&\le A_3^{|\alpha|+1}(|\alpha|!)^s, \quad \text{some } A_3 > 0 \\
&\le A_3^{|\alpha|+1} 2^{s|\alpha|}(\alpha!)^s, \quad \text{since } |\alpha|! \le 2^{|\alpha|}\alpha! \\
&\le A_4^{|\alpha|+1}(\alpha!)^s \quad \text{for some } A_4 > 0.
\end{aligned}$$

Therefore, $h(x) \in G^s(B_r)$ and so the proof is complete.

For $\Gamma \subset \mathbb{R}^m$ a cone and $\delta > 0$, we set

$$\Gamma^\delta = \{v \in \Gamma : |v| < \delta\}.$$

Definition 3 If $V \subset \mathbb{R}^m$ is open, we say a function $f(x, y)$ defined on $V + i\Gamma^\delta$ is of tempered growth if

$$|f(x, y)| \le C|y|^{-k}$$

for some constant C and positive integer k.

The following theorem is a microlocal version of Theorem 2.1.

Theorem 2 *Let $u \in \mathscr{D}'(\Omega)$. Then for any $x_0 \in \Omega$ and $\xi^0 \in \mathbb{R}^m \setminus \{0\}$, $(x_0, \xi^0) \notin WF_s(u)(s > 1)$ if and only if there is a neighborhood V of x_0, acute open cones $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{R}^m \setminus \{0\}$ and C^1 functions f_j on $V + i\Gamma_j^\delta$ (for some $\delta > 0$) of tempered growth such that*

1. $u = \sum_{j=1}^{n} bf_j$ *near* x_0,
2. $\xi^0 \cdot \Gamma_j < 0, \forall j$,
3.

$$\left|\frac{\partial f_j}{\partial \bar{z_k}}(x,y)\right| \le A \exp\left(\frac{-\epsilon}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1, 2, \ldots, n, \forall k = 1, 2, \ldots, m$$

for some $A, \epsilon > 0$.

Equivalently, $\left|\frac{\partial f_j}{\partial \bar{z_k}}(x,y)\right| \le c^{N+1} N^N |y|^{\frac{N}{s-1}}, N = 0, 1, 2, \ldots, .$

Proof Suppose $u = bf$ on V where f is C^1 and of tempered growth on $V + i\Gamma^\delta$, $\xi^0 \cdot \Gamma < 0$ and

$$\left|\frac{\partial f}{\partial \bar{z_j}}(x,y)\right| \le A \exp\left(\frac{-\epsilon}{|y|^{\frac{1}{s-1}}}\right) \quad j = 1, 2, \ldots, m \tag{22}$$

for some $A > 0$, V a neighborhood of x_0 and Γ some conic set. We want to show that $(x_0, \xi^0) \notin WF_s(u)$, $s > 1$. By Corollary 1.4.11 in [13], for each $n \ge 1$, we can choose smooth functions $f_n(x)$ that satisfy

1. $f_n(x) = 1$ on $B_r(0)$, $\text{supp}(f_n) \subset B_{2r}(0)$, for some $r > 0$ and
2. $|D^\alpha f_n| \le C^{|\alpha|}(n+1)^{|\alpha|}$ for $|\alpha| \le n + 1$, for some $C > 0$ independent of n.

Define

$$F_n(x + iy) = \sum_{|\alpha| \le n} \frac{1}{\alpha!} \partial_x^\alpha f_n(x)(iy)^\alpha. \tag{23}$$

Then

$$\begin{aligned}
\left|\frac{\partial F_n}{\partial \overline{z_j}}(x+iy)\right| &= \left|\frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{|\alpha|\le n}\frac{1}{\alpha!}\partial_x^\alpha f_n(x)(iy)^\alpha\right) + \frac{i}{2}\frac{\partial}{\partial y_j}\left(\sum_{|\alpha|\le n}\frac{1}{\alpha!}\partial_x^\alpha f_n(x)(iy)^\alpha\right)\right| \\
&= \left|\frac{1}{2}\sum_{|\alpha|\le n}\frac{1}{\alpha!}\partial_x^{\alpha+e_j} f_n(x)(iy)^\alpha - \frac{1}{2}\sum_{|\alpha|\le n, \alpha_j \ge 1}\frac{\alpha_j}{\alpha!}\partial_x^\alpha f_n(x)(iy)^{\alpha - e_j}\right| \\
&= \left|\frac{1}{2}\sum_{|\alpha| = n}\frac{1}{\alpha!}\partial_x^{\alpha+e_j} f_n(x)(iy)^\alpha\right| \\
&\le (C+1)^{n+1}(n+1)^{n+1}|y|^n \sum_{|\alpha|=n}\frac{1}{\alpha!}
\end{aligned}$$

$$= \frac{m^n}{n!}(C+1)^{n+1}(n+1)^{n+1}|y|^n$$

$$\left(\text{since } m^n = (1+\ldots+1)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!}\right)$$

$$\leq \frac{1}{n!} C_1^{n+1}(n+1)^{n+1}|y|^n, \quad C_1 > 0 \ \text{(for some } C_1 \text{ independent of } n). \tag{24}$$

Fix $y^0 \in \Gamma$. Since $y^0 \cdot \xi^0 < 0$, there is a conic neighborhood Γ_0 of ξ^0 and a constant $c > 0$ such that

$$y^0 \cdot \xi \leq -c|\xi|, \quad \forall\, \xi \in \Gamma_0. \tag{25}$$

For $0 < \lambda < 1$, let

$$D_\lambda = \left\{x + ity^0 : x \in B_{2r}(0),\ \lambda \leq t \leq 1\right\}.$$

We have

$$|F_n(x+iy)| \leq \sum_{|\alpha|\leq n} \frac{C^{|\alpha|}(n+1)^{|\alpha|}}{\alpha!}|y|^{|\alpha|} = \sum_{k=0}^{n} \sum_{|\alpha|=k} \frac{C^k(n+1)^k}{\alpha!}|y|^k$$

$$= \sum_{k=0}^{n} \frac{(mC(n+1)|y|)^k}{k!}$$

$$\leq e^{n+1} \quad \text{(we choose } \delta \text{ and hence } y \text{ small enough).}$$

This estimate on F_n will be used below. Consider the m-form

$$F(x, y, \xi) = e^{-(x+iy)\cdot\xi} F_n(x+iy) f(x+iy) dz$$

for $(x, y) \in D_\lambda, \xi \in \Gamma_0$, where $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $e^{-iz\cdot\xi}$ is holomorphic in z, we have by Stokes theorem

$$\left|\int_{B_{2r}(0)} F(x, \lambda y^0, \xi) dx\right| \leq \int_{B_{2r}(0)} \left|F(x, y^0, \xi)\right| dx$$

$$+ \sum_{j=1}^{m} \int\int_{D_\lambda} \left| e^{-i(x+iy)\cdot\xi} F_n(x+iy) \frac{\partial f}{\partial \overline{z_j}}(x+iy) d\overline{z_j} \wedge dz \right|$$

$$+ \sum_{j=1}^{m} \int\int_{D_\lambda} \left| e^{-i(x+iy)\cdot\xi} f(x+iy) \frac{\partial F_n}{\partial \overline{z_j}}(x+iy) d\overline{z_j} \wedge dz \right|$$

$$= I_0(\xi) + I_1^\lambda(\xi) + I_2^\lambda(\xi) \tag{26}$$

Consider $I_0(\xi)$: For $\xi \in \Gamma_0$,

$$\begin{aligned} I_0(\xi) &= \int_{B_{2r}(0)} \left|F(x, y^0, \xi)\right| dx \\ &= \int_{B_{2r}(0)} \left|e^{-i(x+iy^0)\cdot\xi} F_n(x+iy^0) f(x+iy^0)\right| dx \\ &\le C' C_1 e^{n+1} \int_{B_{2r}(0)} e^{y^0\cdot\xi} dx, \quad C' = \sup_{B_{2r}(0)} |f(x+iy^0)| \\ &\le C'' e^{n+1} e^{-c|\xi|}, \quad \text{by (25)} \\ &\le C_0^{N+1} e^{n+1} N! |\xi|^{-\frac{N}{s}}, \forall\ \xi \in \Gamma_0, \quad N = 0, 1, 2, \ldots \end{aligned} \tag{27}$$

Consider $I_1^\lambda(\xi)$: Putting $y = ty^0$, and using (22) and (25) we have

$$\begin{aligned} I_1^\lambda(\xi) &= \sum_{j=1}^m \int\int_{D_\lambda} \left| e^{-i(x+ity^0)\cdot\xi} F_n(x+ity^0) \frac{\partial f}{\partial \overline{z_j}}(x+ity^0) d\overline{z_j} \wedge dz \right| \\ &\le A' e^{n+1} e^{-ct|\xi|} \exp\left(\frac{-\epsilon}{|ty^0|^{\frac{1}{s-1}}}\right) \sum_{j=1}^m \int\int_{D_\lambda} \left|d\overline{z_j} \wedge dz\right| \\ &\le A'' e^{n+1} e^{-ct|\xi|} \exp\left(\frac{-\epsilon'}{t^{\frac{1}{s-1}}}\right) \\ &\le A'' e^{n+1} \left(\frac{N}{s}\right)^{\frac{N}{s}} e^{-\frac{N}{s}} \frac{1}{(ct|\xi|)^{\frac{N}{s}}} \left[\left(\frac{s-1}{s}\right) N\right]^{\left(\frac{s-1}{s}\right)N} e^{-\left(\frac{s-1}{s}\right)N} \left(\frac{t^{\frac{1}{s-1}}}{\epsilon'}\right)^{\left(\frac{s-1}{s}\right)N} \\ &\le C_2^{N+1} e^{n+1} N^N |\xi|^{-\frac{N}{s}}, \ N = 0, 1, 2, \ldots, \quad \forall\ \xi \in \Gamma_0, \end{aligned} \tag{28}$$

where we used the inequality $e^{-t} \le d^d e^{-d} \frac{1}{t^d}$ (see 1.2.16 in [13]) with $d = \frac{N}{s}$ for $e^{-ct|\xi|}$ and $d = \left(\frac{s-1}{s}\right) N$ for $\exp\left(-\frac{\epsilon'}{t^{\frac{1}{s-1}}}\right)$.

Finally, consider $I_2^\lambda(\xi)$: Since f is of tempered growth, there are a constant $c' > 0$ and an integer $k \ge 1$ such that

$$|f(x+ity^0)| \le \frac{c'}{t^k |y^0|^k}, \quad \forall\ |x| < 2r,\ \lambda \le t \le 1. \tag{29}$$

Using (24), (25) and (29) we have

$$
\begin{aligned}
I_2^\lambda(\xi) &= \sum_{j=1}^{m} \int\int_{D_\lambda} \left| e^{-i(x+ity^0)\cdot\xi} f(x+ity^0) \frac{\partial F_n}{\partial \overline{z_j}}(x+ity^0) d\overline{z_j} \wedge dz \right| \\
&\leq \frac{c'}{t^k |y^0|^k} \frac{1}{n!} e^{-ct|\xi|} C_1^{n+1} (n+1)^{n+1} |ty^0|^{n-1} \\
&\leq \frac{1}{t^k} e^{-ct|\xi|} \frac{1}{n!} C_3^{n+1} (n+1)^{n+1} t^{n-1} \\
&\leq \frac{1}{t^{k+1}} e^{-ct|\xi|} \frac{1}{n!} C_3^{n+1} (n+1)^{n+1} t^{n}
\end{aligned} \tag{30}
$$

Given N, choose n such that 159

$$
\frac{N}{s} + k + 1 \leq n \leq \frac{N+s}{s} + k + 1.
$$

Since $t \leq 1$, (30) becomes 160

$$
\begin{aligned}
I_2^\lambda(\xi) &\leq \frac{1}{t^{k+1}} e^{-ct|\xi|} C_3^{n+1} \frac{(n+1)^{n+1}}{n!} t^n \\
&\leq \frac{1}{t^{k+1}} e^{-ct|\xi|} C_3^{\frac{N+s}{s}+k+2} (n+1)^{n+1} t^{\frac{N}{s}+k+1} \\
&\leq \left(\frac{N}{s}\right)^{\frac{N}{s}} e^{-\frac{N}{s}} \frac{1}{t^{\frac{N}{s}} c^{\frac{N}{s}} |\xi|^{\frac{N}{s}}} C_4^{\frac{N+s}{s}+k+2} \left(\frac{N+s}{s} + k + 2\right)^{\frac{N+s}{s}+k+2} t^{\frac{N}{s}} \\
&\quad \left(\text{we used } e^{-t} \leq d^d e^{-d} t^{-d} \text{ with } d = \frac{N}{s}\right) \\
&\leq \left(\frac{N}{s}\right)^{\frac{N}{s}} \frac{1}{c^{\frac{N}{s}} |\xi|^{\frac{N}{s}}} C_4^{\frac{N+s}{s}+k+2} \left(\frac{N+s}{s} + k + 2\right)^{\frac{N+s}{s}+k+2} \\
&\leq B^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad \text{some } B > 0,\ N = 0, 1, 2, \ldots, \xi \in \Gamma_0.
\end{aligned} \tag{31}
$$

where B is independent of n. Using (25), (26), (27), (28) and (31), there is a constant 161
$B_1 > 0$ independent of λ such that 162

$$
\begin{aligned}
\left|\widehat{f_n u}(\xi)\right| &= \left| \int_{B_{2r}(0)} e^{-ix\cdot\xi} f_n(x) u(x) dx \right| \\
&= \lim_{\lambda \to 0} \left| \int_{B_{2r}(0)} e^{-i(x+i\lambda y^0)\cdot\xi} F_n(x+i\lambda y^0) f(x+i\lambda y^0) dx \right| \\
&\leq B_1^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad N = 0, 1, 2, \ldots, \xi \in \Gamma_0.
\end{aligned}
$$

Therefore, $(x_0, \xi^0) \notin WF_s(u)$. 163

Conversely, suppose $(x_0, \xi^0) \notin WF_s(u)$. Then there is $\phi \in G^s \cap C_0^\infty$, $\phi \equiv 1$ near x_0 such that

$$\left|\widehat{\phi u}(\xi)\right| \leq C^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad N = 0, 1, 2, \ldots,$$

for ξ in some conic neighborhood Γ of ξ^0 and for some constant $C > 0$. Let C_j, $1 \leq j \leq n$ be acute, open cones such that

$$\mathbb{R}^m = \bigcup_{j=1}^{n} \overline{C_j}, \quad |\overline{C_j} \cap \overline{C_k}| = 0, \ \ j \neq k.$$

Assume that $\xi^0 \in C_1$ and $\xi^0 \notin \overline{C_j}$ for $j \geq 2$. Then we can get acute, open cones $\Gamma_j, 2 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma_j < 0 \quad \text{and} \quad y \cdot \xi \geq c|y||\xi|, \quad \forall\, y \in \Gamma_j, \quad \forall\, \xi \in C_j. \tag{32}$$

By the inversion formula we have

$$\phi(x)u(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{ix\cdot\xi} \widehat{\phi u}(\xi) d\xi = \frac{1}{(2\pi)^m} \sum_{j=1}^{n} \int_{C_j} e^{ix\cdot\xi} \widehat{\phi u}(\xi) d\xi.$$

For $x + iy \in \mathbb{R}^m + i\Gamma_j, j \geq 2$ define

$$f_j(x+iy) = \int_{C_j} e^{i(x+iy)\cdot\xi} \widehat{\phi u}(\xi) \frac{d\xi}{(2\pi)^m}.$$

using (32), we see that f_j $(j \geq 2)$ is holomorphic on the wedge $\mathbb{R}^m + i\Gamma_j$ and is of tempered growth. Let

$$g_1(x) = \int_{C_1} e^{ix\cdot\xi} \widehat{\phi u}(\xi) \frac{d\xi}{(2\pi)^m} = g_{11}(x) + g_{12}(x)$$

where

$$g_{11}(x) = \int_{\xi\in C_1, |\xi|\leq 1} e^{ix\cdot\xi} \widehat{\phi u}(\xi) \frac{d\xi}{(2\pi)^m}, \quad g_{12}(x) = \int_{\xi\in C_1, |\xi|\geq 1} e^{ix\cdot\xi} \widehat{\phi u}(\xi) \frac{d\xi}{(2\pi)^m}.$$

Assume $C_1 \subset \Gamma$. Clearly $g_{11}(x)$ is real analytic on $\mathbb{R}^m$. We have

$$\begin{aligned}
|\partial^\alpha g_{12}(x)| &= \left| \int_{\xi\in C_1, |\xi|\geq 1} e^{ix\cdot\xi} \xi^\alpha \widehat{\phi u}(\xi) \frac{d\xi}{(2\pi)^m} \right| \\
&\leq C^{N+1} N! \int_{\xi\in C_1, |\xi|\geq 1} |\xi|^{|\alpha|} |\xi|^{-\frac{N}{s}} \, d\xi
\end{aligned}$$

$$\leq C^{N+1}N^N \int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}d\xi$$

$$\leq C_2^{(m+1+|\alpha|)s+1}\,[(m+1+|\alpha|)s]^{(m+1+|\alpha|)s} \int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-m-1-|\alpha|}d\xi$$

$$\text{(taking } N \backsim (m+1+|\alpha|)s)$$

$$\leq A^{|\alpha|+1}(\alpha!)^s, \quad \text{for some } A>0.$$

Therefore, $g_1 \in G^s$. By theorem 1, if K is a compact set whose interior contains x_0, there is $f_1(x+iy) \in C^1(K+i\mathbb{R}^m)$ such that $f_1(x) = g_1(x), x \in K$ and

$$\left|\frac{\partial f_1}{\partial \bar{z}_j}(x,y)\right| \leq c_1 \exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1,2,\dots,m$$

for some constants $c_1, c_2 > 0$. Let Γ_1 be any open cone such that $\xi^0\cdot\Gamma_1 < 0$. Let $V \subset K$ be an open such that $x_0 \in V$. Then we have found functions $f_j(x+iy)(1 \leq j \leq n)$ C^1 on $V + i\Gamma_j^\delta$ (for some $\delta > 0$) and of tempered growth such that $\phi u = \sum_{j=1}^n bf_j$ on V. By contracting V we have $\phi \equiv 1$ on V and so $u = \sum_{j=1}^n bf_j$ on V. Thus, the proof is complete.

3 Characterization of the Gevrey Wave Front Set

For $u \in \mathscr{E}'(\mathbb{R}^m)$ we recall that the classical FBI transform of u is

$$\mathscr{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^2}u(x')dx'.$$

We recall the following theorem of M. Christ which characterizes the Gevrey wave front set of a function in terms of the classical FBI transform.

Theorem 3 *([7]). Let $u \in \mathscr{E}'(\mathbb{R}^m)$. Let $x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then $(x_0,\xi^0) \notin WF_s(u)$ if and only if there is a neighborhood V of x_0, a conic neighborhood Γ of ξ^0 such that for some $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi \equiv 1$ near x_0,*

$$|\mathscr{F}(\varphi u)(x,\xi)| \leq c_1 \exp\left(-c_2|\xi|^{\frac{1}{s}}\right), \forall (x,\xi) \in V \times \Gamma$$

for some constants $c_1, c_2 > 0$.

Our goal is to generalize Christ's theorem to a subclass of the generalized FBI transforms introduced in [6]. We will consider a polynomial which is a sum of elliptic, homogeneous polynomials.

Let $p(x)$ be a positive polynomial of the form

$$p(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha + \sum_{|\beta|=2k} b_\beta x^\beta, a_\alpha, b_\beta \in \mathbb{R}, l \neq k$$

which satisfies

$$c_1|x|^{2l} \leq \sum_{|\alpha|=2l} a_\alpha x^\alpha \leq c_2|x|^{2l}$$

and

$$c_3|x|^{2k} \leq \sum_{|\beta|=2k} b_\beta x^\beta \leq c_4|x|^{2k}$$

for some constants $0 < c_1 \leq c_2$ and $0 < c_3 \leq c_4$.

Suppose $l < k$ and let

$$p_1(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha, p_2(x) = \sum_{|\beta|=2k} b_\beta x^\beta.$$

Take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter. Let $c_p > 0$ be a constant such that

$$c_p \int_{\mathbb{R}^m} \psi(x) dx = 1.$$

In this section we will consider the FBI transform given by

$$\begin{aligned}\mathscr{F}u(t,\xi) &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')} \psi(|\xi|^\lambda (t-x'))u(x')dx' \\ &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')-|\xi|^{\frac{l}{k}} p_1(t-x')-|\xi| p_2(t-x')} u(x')dx'.\end{aligned}$$

Let $\chi(x) \in S(\mathbb{R}^m)$ such that $\int_{\mathbb{R}^m} \chi(x)dx = 1$. Set

$$\sigma(\xi) = \frac{\hat{\chi}(\xi)}{(2\pi)^m}.$$

Then the inversion formula becomes

$$u(x) = \lim_{\epsilon\to 0+} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi) \mathscr{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi.$$

We will show that this class of FBI transforms characterizes the Gevrey wave front set of a distribution. We remark that the proof will also work for $p(x)$ that is a sum of a finite number of elliptic, homogeneous polynomials.

Theorem 4 *Let* $u \in \mathscr{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$ *with* $|\xi^0| = 1$. *Then* $(x_0, \xi^0) \notin WF_s(u), s > 1$ *if and only if there exist a neighborhood* V *of* x_0 *, a conic neighborhood* Γ *of* ξ^0 *and constants* $a, b > 0$ *such that for some* $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ *near* x_0,

$$|\mathscr{F}(\phi u)(t,\xi)| \leq a e^{-b|\xi|^{\frac{1}{s}}},\ (t,\xi) \in V \times \Gamma.$$

Proof Suppose $(x_0, \xi^0) \notin WF_s(u)$. We may assume that $x_0 = 0$. By Theorem 2.3, without loss of generality, there is f which is C^1 in some truncated wedge $V + i\Gamma_\delta$ (for some $\delta > 0$) and of tempered growth with V a neighborhood of 0 and Γ an open cone such that

1. $u = bf$ on V,
2. $\xi^0 \cdot \Gamma < 0$, and
3. $$\left|\frac{\partial f}{\partial \bar{z}_j}(x+iy)\right| \leq A \exp\left(\frac{-B}{|y|^{\frac{1}{s-1}}}\right),\ x+iy \in V + i\Gamma_\delta$$

   for some $A, B > 0$.

Let $r > 0$ such that

$$B_{2r} = \{x : |x| < 2r\} \subset\subset V.$$

Let $\phi(x) \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ on B_r and $\text{supp}(\phi) \subset B_{2r}$.

Fix $v \in \Gamma_\delta$.

Let

$$Q(x', \xi, x) = i\xi \cdot (x'-x) - |\xi|^{\frac{l}{k}} p_1(x'-x) - |\xi| p_2(x'-x).$$

Then

$$\begin{aligned}\mathscr{F}(\phi u)(x', \xi) &= c_p \int_{\mathbb{R}^m} e^{Q(x',\xi,x)} \phi(x) u(x) dx \\ &= c_p \left\langle bf, \phi(x) e^{Q(x',\xi,x)} \right\rangle \\ &= c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x)} \phi(x) f(x+itv) dx.\end{aligned}$$

Since $\phi(x) \in C^\infty(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x+iy)$ smooth on $V + i\mathbb{R}^m$ with $x-$ support in B_{2r}. Then 223 224

$$\mathscr{F}(\phi u)(x', \xi) = c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x+itv) f(x+itv) dx.$$

For $0 < \lambda < 1$, let 225

$$D_\lambda = \{x + itv \in \mathbb{C}^m : x \in B_{2r},\ \lambda \le t \le 1\}.$$

Consider the m-form 226

$$\omega(z) = e^{Q(x',\xi,z)} \tilde{\phi}(z) f(z) dz_1 \wedge \ldots \wedge dz_m, z = x + iy.$$

Let $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $\tilde{\phi}(x+iy) = 0$ for $|x| \ge 2r$ and since $e^{Q(x',\xi,z)}$ is holomorphic in z, by Stokes theorem 227 228

$$\begin{aligned}
\mathscr{F}(\phi u)(x', \xi) &= c_p \lim_{\lambda \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+i\lambda v)} \tilde{\phi}(x+i\lambda v) f(x+i\lambda v) dx \\
&= c_p \int_{B_{2r}} e^{Q(x',\xi,x+iv)} \tilde{\phi}(x+iv) f(x+iv) dx \\
&+ c_p \lim_{\lambda \to 0+} \sum_{j=1}^m \int \int_{D_\lambda} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x+itv) \frac{\partial f}{\partial \bar{z}_j}(x+itv) d\bar{z}_j \wedge dz \\
&+ c_p \lim_{\lambda \to 0+} \sum_{j=1}^m \int \int_{D_\lambda} e^{Q(x',\xi,x+itv)} \frac{\partial \tilde{\phi}}{\partial \bar{z}_j}(x+itv) f(x+itv) d\bar{z}_j \wedge dz \\
&= I_0(x',\xi) + \lim_{\lambda \to 0+} \left(I_1^\lambda(x',\xi) + I_2^\lambda(x',\xi) \right)
\end{aligned}$$

Since $v \in \Gamma$ and $\xi^0 \cdot \Gamma < 0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that 229 230

$$\xi \cdot v \le -c|\xi||v|, \ \forall \xi \in \Gamma_1.$$

Consider $I_0(x', \xi)$: 231

$$|I_0(x',\xi)| \le \sup_{x \in \overline{B_{2r}}} |c_p \tilde{\phi}(x+iv) f(x+iv)| \int_{B_{2r}} e^{\Re Q(x',\xi,x+iv)} dx.$$

For $\xi \in \Gamma_1$, $|\xi| \ge 1$, since $l < k$, 232

$$\begin{aligned}
&\Re Q(x', \xi, x+iv) \\
&= \Re \left(i\xi \cdot (x' - x - iv) - |\xi|^{\frac{l}{k}} p_1(x' - x - iv) - |\xi| p_2(x' - x - iv) \right)
\end{aligned}$$

$$= \xi \cdot v - |\xi|^{\frac{l}{k}} \Re p_1(x' - x - iv) - |\xi| \Re p_2(x' - x - iv)$$
$$= \xi \cdot v - |\xi|^{\frac{l}{k}} p_1(x' - x) + O(|v|^2)|\xi|^{\frac{l}{k}} - |\xi| p_2(x' - x) + O(|v|^2)|\xi|$$
$$\leq -c|v||\xi| - c_1|\xi|^{\frac{l}{k}}|x' - x|^{2l}$$
$$+ O(|v|^2)|\xi|^{\frac{l}{k}} - c_3|\xi||x' - x|^{2k} + O(|v|^2)|\xi|$$
$$\leq -c|v||\xi| + O(|v|^2)|\xi|$$

choosing $|v|$ small such that $O(|v|^2) \leq \frac{c|v|}{2} = c'$. Then 233

$$\Re Q(x', \xi, x + iv) \leq -c'|\xi|, \xi \in \Gamma_1, |\xi| \geq 1, x' \in \mathbb{R}^m.$$

Thus, for $\xi \in \Gamma_1, |\xi| \geq 1$, 234

$$|I_0(x', \xi)| \leq c'' e^{-c'|\xi|} \leq c'' e^{-c'|\xi|^{\frac{1}{s}}}$$

for some $c'' > 0$. Since 235

$$\frac{I_0(x', \xi)}{e^{-c'|\xi|^{\frac{1}{s}}}}$$

is bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$, there are $A_0, B_0 > 0$ such that 236

$$|I_0(x', \xi)| \leq A_0 e^{-B_0|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, |x'| < 2r. \quad (33)$$

Consider 237

$$I_1^\lambda(x', \xi) = c_p \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x', \xi, x+itv)} \tilde{\phi}(x + itv) \frac{\partial f}{\partial \bar{z}_j}(x + itv) d\bar{z}_j \wedge dz :$$

For $\xi \in \Gamma_1, |\xi| \geq 1$, 238

$$\left| e^{Q(x', \xi, x+itv)} \tilde{\phi}(x + itv) \frac{\partial f}{\partial \bar{z}_j}(x + itv) \right|$$
$$\leq C' e^{\Re Q(x', \xi, x+itv)} A \exp\left(\frac{-B}{|tv|^{\frac{1}{s-1}}} \right), \quad C' = \sup_{(x,t) \in \overline{B_{2r}} \times [0,1]} \left| \tilde{\phi}(x + itv) \right|$$
$$\leq A' e^{-ct|v||\xi| - c_1|x' - x|^{2k}|\xi| + O(|tv|^2)|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}} \right)$$
$$\leq A' e^{-ct|v||\xi| - c_1|x' - x|^{2k}|\xi| + A'' t^2|v|^2|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}} \right), \quad \text{some } A' > 0$$

$$
\begin{aligned}
&\le A' e^{-ct|v||\xi| + A'' t|v|^2|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \\
&\le A' e^{-c' t|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \ \text{(take } |v| \text{ small such that } A''|v|^2 \le \tfrac{c|v|}{2} = c'\text{)} \\
&\le C^{N+1} N! |\xi|^{\frac{-N}{s}}, \text{some } C > 0, N = 0, 1, 2, \ldots,
\end{aligned}
$$

where we used the inequality

$$e^{-\alpha} \le d^d e^{-d} \alpha^{-d}, \ \ d, \alpha > 0$$

with $d = \frac{N}{s}$ for $e^{-c' t|\xi|}$, and $d = \frac{(s-1)}{s} N$ for $\exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right)$ for $N \ge 1$.

Hence

$$
\begin{aligned}
&\lim_{\lambda \to 0+} |I_1^\lambda(x', \xi)| \\
&= c_p \lim_{\lambda \to 0+} \left| \sum_{j=1}^m \int\int_{D_\lambda} e^{Q(x', \xi, x+itv)} \tilde{\phi}(x+itv) \frac{\partial f}{\partial \overline{z_j}}(x+itv) d\overline{z_j} \wedge dz \right| \\
&\le \lim_{\lambda \to 0+} C^{N+1} N^N |\xi|^{\frac{-N}{s}} \sum_{j=1}^m \int_0^1 \int_{B_{2r}} d\bar{z}_j \wedge dz \\
&\le D^{N+1} N! |\xi|^{\frac{-N}{s}}, \xi \in \Gamma_1, |\xi| \ge 1, x' \in \mathbb{R}^m, \text{some } D > 0.
\end{aligned}
$$

Therefore,

$$\lim_{\lambda \to 0+} |I_1^\lambda(x', \xi)| \le a_1 \exp\left(-b_1 |\xi|^{\frac{1}{s}}\right), \forall \xi \in \Gamma_1, |\xi| \ge 1, x' \in B_{2r}$$

for some $a_1, b_1 > 0$ independent of λ. But

$$\frac{|I_1^\lambda(x', \xi)|}{\exp(-b_1 |\xi|^{\frac{1}{s}})}$$

is uniformly bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \le 1\}$. Thus, there are $A_1, B_1 > 0$ such that

$$\lim_{\lambda \to 0+} |I_1^\lambda(x', \xi)| \le A_1 \exp\left(-B_1 |\xi|^{\frac{1}{s}}\right), \forall \xi \in \Gamma_1, |x'| < 2. \tag{34}$$

Consider

$$I_2^\lambda(x', \xi) = \sum_{j=1}^m \int\int_{D_\lambda} e^{Q(x', \xi, x+itv)} \frac{\partial \tilde{\phi}}{\partial \overline{z_j}}(x+itv) f(x+itv) d\overline{z_j} \wedge dz :$$

For $\xi \in \Gamma_1, |\xi| \geq 1,$ 246

$$\begin{aligned}\Re Q(x', \xi, x + itv) &\leq -ct|v||\xi| + O(t^2|v|^2)|\xi| - c_3|\xi||x' - x|^{2k} \\ &\leq O(|v|^2)|\xi| - c_3|\xi||x' - x|^{2k} \quad \text{since } t \leq 1 \\ &\leq a'|v|^2|\xi| - c_3|\xi||x' - x|^{2k}.\end{aligned}$$

Since $\frac{\partial \tilde{\phi}}{\partial \overline{z_j}} \equiv 0$ for $|x| \leq r$, the integral over $|x| \leq r$ is zero. Then for $|x'| < \frac{r}{2}$ and 247
$|x| \geq r$, 248

$$\Re Q(x', \xi, x + itv) \leq a'|v|^2|\xi| - c_1 \frac{r^{2k}}{2^{2k}}|\xi|.$$

Choose $|v|$ small such that 249

$$a'|v|^2 \leq c_1 \frac{r^{2k}}{2^{2k+1}} = c''.$$

We then get 250

$$\Re Q(x', \xi, x + itv) \leq -c''|\xi|, \quad \xi \in \Gamma_1, \ |\xi| \geq 1.$$

Since f is of tempered growth, there is a constant $d > 0$ and an integer $n \geq 0$ 251
such that 252

$$|f(x + itv)| \leq \frac{d}{t^n|v|^n}.$$

Since $\tilde{\phi}$ is almost holomorphic, there is $c_n > 0$ such that 253

$$\left|\frac{\partial \tilde{\phi}}{\partial \overline{z_j}}(x + itv)\right| \leq c_n t^n |v|^n \quad \forall j = 1, 2, \ldots, m.$$

Thus we can get $A_2, B_2 > 0$ independent of λ such that 254

$$\lim_{\lambda \to 0+} |I_2^{\lambda}(x', \xi)| \leq A_2 e^{-B_2|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, \ |x'| < \frac{r}{2}. \tag{35}$$

Therefore, from (3.1), (3.2), and (3.3), we can find constants $A, B > 0$ such that 255

$$|\mathscr{F}(\phi u)(x', \xi)| \leq A e^{-B|\xi|^{\frac{1}{s}}}, \ \forall (x', \xi) \in B_{\frac{r}{2}} \times \Gamma_1$$

where Γ_1 is a conic neighborhood of ξ^0. 256

Conversely, suppose

$$|\mathscr{F}(\phi u)(t,\xi)| \le c_1 e^{-c_2|\xi|^{\frac{1}{s}}}, (t,\xi) \in V \times \Gamma$$

where V is some neighborhood of 0, Γ a conic neighborhood of ξ^0, and $c_1, c_2 > 0$ are some constants and $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near 0.

We want to show that $(0,\xi^0) \notin WF_s(u)$. Let $\sigma(\xi) = e^{-|\xi|^2}$. We apply the inversion formula

$$\phi(x)u(x) = \lim_{\epsilon\to 0+} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)-\epsilon^2|\xi|^2} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

Let

$$u_\epsilon(z) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi,\ z = x+iy \in \mathbb{C}^m.$$

Clearly $u_\epsilon(z)$ is an entire function of z for each $\epsilon > 0$.

We write

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z)$$

where for some $a > 0$ we set

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|t|\le a} e^{i\xi\cdot(z-t)} \sigma(\epsilon\xi) \mathscr{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt\, d\xi$$

and

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|t|\ge a} e^{i\xi\cdot(z-t)} \sigma(\epsilon\xi) \mathscr{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt\, d\xi.$$

Consider $u_0^\epsilon(z)$: Choose $a > 0$ such that $\{t : |t| \le a\} \subset V$. Let $\mathscr{C}_0 = \Gamma, \mathscr{C}_j, 1 \le j \le n$ be open acute cones (we may take Γ to be acute) such that $\mathbb{R}^m = \bigcup_{j=0}^n \overline{\mathscr{C}_j}$, $\overline{\mathscr{C}_j} \cap \overline{\mathscr{C}_k}$ has measure zero when $j \ne k$ and $\xi^0 \notin \overline{\mathscr{C}_j}$ for $j \ge 1$.

Since $\xi^0 \notin \overline{\mathscr{C}_j}$ and $\mathscr{C}_j$ is acute we can get acute, open cones $\Gamma^j, 1 \le j \le n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \ \text{ and } y\cdot\xi \ge c|y||\xi|, \forall y \in \Gamma^j, \forall \xi \in \mathscr{C}_j.$$

We have

$$u_0^\epsilon(x) = \sum_{j=0}^n \int_{\mathscr{C}_j} \int_{|t|\le a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi = \sum_{j=0}^n v_j^\epsilon(x).$$

For $j = 0, 1, \dots, n$, and $z = x + iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^{\epsilon}(x+iy) = \int_{\mathscr{C}_j} \int_{|t|\le a} e^{i\xi\cdot(x+iy-t)-\epsilon|\xi|^2} \mathscr{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

$f_j^{\epsilon}(z)$ are entire for $j \ge 1$ and converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathscr{C}_j} \int_{|t|\le a} e^{i\xi\cdot(x+iy-t)} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi$$

which is holomorphic and of tempered growth on $\mathbb{R}^m + i\Gamma_\delta^j$ for some $0 < \delta \le 1$. Thus each $f_j, j = 1, \dots, n$ has a boundary value $bf_j \in \mathscr{D}'(\mathbb{R}^m)$.

Let

$$g_0^{\epsilon}(x) = \int_{\Gamma} \int_{|t|\le a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

By the estimate for $\mathscr{F}(\phi u)(t,\xi)$ on the set $\{t : |t| \le a\} \times \Gamma$, $g_0^{\epsilon}(x)$ are smooth for all $\epsilon > 0$ and converge uniformly on $\mathbb{R}^m$ to the function

$$g_0(x) = \int_{\Gamma} \int_{|t|\le a} e^{i\xi\cdot(x-t)} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

Clearly $g_0(x)$ is smooth on $\mathbb{R}^m$.

For any α,

$$\begin{aligned}
|\partial^\alpha g_0(x)| &= \left| \int_{\Gamma} \int_{|t|\le a} \xi^\alpha e^{i\xi\cdot(x-t)} \mathscr{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi \right| \\
&\le d_1 \int_{\Gamma} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} |\xi|^{\frac{m}{2k}} d\xi, d_1 > 0 \\
&\le d_1 \int_{|\xi|\le 1} d\xi + d_1 \int_{\xi\in\Gamma, |\xi|\ge 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} |\xi|^m d\xi \\
&= d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} \int_{\xi\in\Gamma, |\xi|\ge 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} \left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{ms} d\xi, d_2 > 0 \\
&\le d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} \int_{\xi\in\Gamma, |\xi|\ge 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} \left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{N'} d\xi \\
&\qquad \left(N' = \min\{N \in \mathbb{N} : N \ge ms\}\right) \\
&\le d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} N'! \int_{\xi\in\Gamma, |\xi|\ge 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} e^{\frac{c_2}{2}|\xi|^{\frac{1}{s}}} d\xi
\end{aligned}$$

$$\leq d_2 + d_3 \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} e^{\frac{-c_2}{2}|\xi|^{\frac{1}{s}}} d\xi \quad (\text{some } d_3 > 0)$$

$$\leq d_2 + d_3 \left(\frac{2}{c_2}\right)^N N! \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{\frac{-N}{s}} d\xi, \forall N = 1, 2, \ldots$$

$$\leq d_2 + d^N N^N \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{\frac{-N}{s}} d\xi, \quad (\text{since } N! \leq N^N)$$

$$\leq d_2 + d_4^{(m+|\alpha|+1)s}(m + |\alpha| + 1)^{(m+|\alpha|+1)s}$$

$$(\text{taking } N \text{ such that } (m + |\alpha|)s \leq N \leq (m + |\alpha| + 1)s)$$

$$\leq d_2 + (ed_4)^{(m+|\alpha|+1)s}((m + |\alpha| + 1)!)^s \quad \text{since } n^n \leq e^n\, n!$$

$$\leq d_2 + (2ed_4)^{(m+|\alpha|+1)s}[(m + 1)!]^s(|\alpha|!)^s \qquad (\text{we used } (j + k)! \leq 2^{k+j}k!j!)$$

$$\leq F^{|\alpha|+1}(\alpha!)^s \quad \text{since } |\alpha|! \leq 2^{|\alpha|}\alpha!$$

for some $F > 0$ independent of α. Hence $g_0 \in G^s(\mathbb{R}^m)$. Thus there is $f_0(x, y) \in$ 284
$C^1(V \times \mathbb{R}^m)$ such that $f_0(x, 0) = g_0(x)$ and 285

$$\left|\frac{\partial f_0}{\partial \bar{z}_j}(x, y)\right| \leq A_1 \left(\frac{-A_2}{|y|^{\frac{1}{s-1}}}\right).$$

Choose Γ_0 an open cone such that $\xi^0 \cdot \Gamma_0 < 0$. Thus we have found open cones 286
$\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and functions f_j holomorphic on $\mathbb{R}^m + i\Gamma_j^\delta$ (for some $\delta > 0$) for 287
$j \geq 1$ which are of tempered growth and $f_0(x, y)$ smooth and of tempered growth on 288
$\mathbb{R}^m + i\Gamma_0^\delta$ (for some $\delta > 0$) such that 289

$$\xi^0 \cdot \Gamma_j < 0, \ \ 0 \leq j \leq n$$

and 290

$$\left|\frac{\partial f_j}{\partial \bar{z}_k}(x, y)\right| \leq A_1 \left(\frac{-A_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1, 2 \ldots, n, \forall k = 0, 1, 2, \ldots m.$$

It is readily seen that in the sense of distributions, for all $j = 1, \ldots, n$, 291

$$\lim_{\Gamma_j \ni y\to 0} f_j(x + iy) = \lim_{\epsilon\to 0+} f_j^\epsilon(x)$$

and 292

$$\lim_{\Gamma_0 \ni y\to 0} f_0(x + iy) = \lim_{\epsilon\to 0+} g_0^\epsilon(x).$$

Hence 293

$$u_0(x) = \sum_{j=0}^{n} bf_j$$

in $\mathscr{D}'(\mathbb{R}^m)$. By Theorem 2.3, we conclude that $(0, \xi^0) \notin WF_s(u_0)$. 294

Consider $u_1^\epsilon(z)$: We will show that $(u_1^\epsilon(z))$ is uniformly bounded for z near 0. Write

$$u_1^\epsilon(z) = \sum_{j=1}^{3} I_j^\epsilon(z)$$

where for some $A > 0$ to be chosen later

$$\begin{aligned} I_1^\epsilon(z) = &\quad \text{the integral over } X_1 = \{(t,\xi) : a \le |t| \le A, |\xi| \le 1\} \\ I_2^\epsilon(z) = &\quad \text{the integral over } X_2 = \{(t,\xi) : |t| \ge A, \xi \in \mathbb{R}^m\} \\ I_3^\epsilon(z) = &\quad \text{the integral over } X_3 = \{(t,\xi) : a \le |t| \le A, |\xi| \ge 1\} \end{aligned}$$

Since X_1 is a bounded set and $\mathscr{F}(\phi u)$ is continuous function it is clear that there is a constant $C_1 > 0$ independent of $0 < \epsilon \le 1$ such that

$$|I_1^\epsilon(z)| \le \int_{X_1} e^{-y\cdot\xi - \epsilon|\xi|^2} |\mathscr{F}(\phi u)(t,\xi)| |\xi|^{\frac{m}{2k}} dt d\xi \le C_1, \forall |y| < 1. \tag{36}$$

Consider $I_2^\epsilon(z)$: Let $r > 0$ such that

$$\text{supp}(\phi) \subset \{x : |x| \le r\} = B_r.$$

Choose $A = 2r$. Then for $|x'| \le r$ and $|t| \ge A$,

$$|t - x'| \ge \frac{|t|}{4} + \frac{A}{4}$$

and so

$$|t - x'|^{2k} \ge \frac{|t|^{2k}}{4^{2k}} + \frac{A^{2k}}{4^{2k}}.$$

We have

$$\begin{aligned} |\mathscr{F}(\phi u)(t,\xi)| &= \left| \int_{|x'| \le r} e^{i\xi\cdot(t-x')} \psi(|\xi|^{\frac{1}{2k}}(t-x')) \phi(x') u(x') dx' \right| \\ &= \left| \int_{|x'| \le r} e^{i\xi\cdot(t-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x')} \phi(x') u(x') dx' \right| \\ &\le C \sup_{|x'| \le r, |\alpha| \le N_1} \left| \partial_{x'}^\alpha \left(e^{i\xi\cdot(t-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x')} \right) \right|, \quad N_1 = \text{the order of } u \end{aligned}$$

To estimate the preceding expression, we observe that if c is a constant and $A(x)$ is a smooth function, for any multi-index β, the derivative $\partial_x^{\beta} e^{cA(x)}$ is a sum of terms of the form $c^{l_1+\cdots+l_n}(\partial^{m_1}p)^{l_1}\cdots(\partial^{m_n}p)^{l_n}$ where $\sum_{j=1}^{n} m_j l_j = |\beta|$. This observation together with the fact that $e^{-c} \le \frac{k!}{c^k}$ for any $c > 0$ leads to

$$|\mathscr{F}(\phi u)(t,\xi)| \le C' e^{-A_1|\xi||t|^{2k} - B_1|\xi|}, |t| \ge A, \xi \in \mathbb{R}^m$$

for some constants $C', A_1, B_1 > 0$ independent of $\epsilon > 0$. Therefore,

$$\begin{aligned}
|I_2^{\epsilon}(z)| &= \left| \int_{\mathbb{R}^m} \int_{|t|\ge A} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2} \mathscr{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}}\, dt d\xi \right| \\
&\le C' \int_{\mathbb{R}^m} \int_{|t|\ge A} e^{|y||\xi|} e^{-A_1|\xi||t|^{2k} - B_1|\xi|} |\xi|^{\frac{m}{2k}}\, dt d\xi \\
&= C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{|t|\ge A} e^{-A_1|\xi||t|^{2k}} dt \right) d\xi \\
&= C'' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} \\
&\le C'' \int_{\mathbb{R}^m} e^{\frac{-B_1}{2}|\xi|} d\xi, \ \ \forall z = x + iy, \ \ |y| < \frac{B_1}{2}.
\end{aligned}$$

It follows that there is $C_2 > 0$ independent of $0 < \epsilon \le 1$ such that

$$|I_2^{\epsilon}(z)| \le C_2, \forall |z| < \delta_2 = \frac{b-1}{2}, \ \ \forall\ 0 < \epsilon \le 1.$$

Consider $I_3^{\epsilon}(z)$:

$$I_3^{\epsilon}(z) = \int\!\!\int\!\!\int_R e^{i\xi\cdot(z-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x') - \epsilon|\xi|^2} \phi(x') u(x') |\xi|^{\frac{m}{2k}}\, d\xi dx' dt$$

where

$$R = \left\{(\xi, x', t) :\ |\xi| \ge 1, |x'| \le r, a \le |t| \le A\right\}$$

Using a branch of the logarithm we note that the function $\xi \mapsto |\xi|$ has a holomorphic extension

$$\langle\zeta\rangle = \left(\sum_{j=1}^{m} \zeta_j^2\right)^{\frac{1}{2}}.$$

In particular, the functions $\zeta \mapsto \langle\zeta\rangle$ and $\zeta \mapsto \langle\zeta\rangle^{\frac{m}{2k}}$ are holomorphic on the set

$$S = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\eta| < |\xi|\}.$$

Fix x, x'. Then we will change the contour of integration in ξ from the m-cycle $\{\xi : |\xi| \geq 1\} \subset \mathbb{R}^m$ to its image under the map

$$\zeta(\xi) = \xi + ib|\xi|(x - x')$$

where $b > 0$ is chosen small so that

$$|\Im\zeta(\xi)| = b|\xi||x - x'| < |\Re\zeta(\xi)| = |\xi|$$

Let

$$D = \left\{\xi + i\sigma b|\xi|(x - x') : |\xi| \geq 1, 0 \leq \sigma \leq 1\right\}.$$

Consider the m-form

$$\omega(z, x', t, \zeta, \epsilon) = e^{i(z-x')\cdot\zeta - \langle\zeta\rangle^{\frac{l}{k}} p_1(t-x') - \langle\zeta\rangle p_2(t-x') - \epsilon\langle\zeta\rangle^2} \phi(x')u(x')\langle\zeta\rangle^{\frac{m}{2k}} d\zeta$$

where $\zeta = \xi + i\eta \in \mathbb{C}^m$, $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_m$. Since

$$g(\zeta) = e^{i(z-x')\cdot\zeta - \langle\zeta\rangle^{\frac{l}{k}} p_1(t-x') - \langle\zeta\rangle p_2(t-x') - \epsilon\langle\zeta\rangle^2} \phi(x')u(x')\langle\zeta\rangle^{\frac{m}{2k}}$$

is a holomorphic function of ζ, ω is a closed form. So by Stokes theorem

$$\int_{\partial D} \omega d\zeta = \int_D d\omega \wedge d\zeta = 0.$$

Now

$$\partial D = \{\xi : |\xi| \geq 1\} \bigcup \left\{\xi + ib|\xi|(x - x') : |\xi| \geq 1\right\}$$
$$\bigcup \left\{\xi + i\sigma b|\xi|(x - x') : |\xi| = 1, 0 \leq \sigma \leq 1\right\}.$$

Therefore,

$$\begin{aligned}
&\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x') - \epsilon|\xi|^2} \phi(x')u(x')|\xi|^{\frac{m}{2k}} d\xi \\
&= \int_{|\xi|\geq 1} \omega(z, x', \xi + ib|\xi|(x - x'))d\xi \\
&- \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x'))d\xi d\sigma
\end{aligned}$$

Clearly there is $B_1 > 0$ independent of ϵ such that

$$\left| \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x'))d\xi d\sigma \right| \leq B_1.$$

To estimate the other integrals, let

$$Q(z, x', t, \xi, \epsilon) = i(z - x') \cdot \zeta(\xi) - \langle \zeta(\xi) \rangle^{\frac{l}{k}} p_1(t - x') - \langle \zeta(\xi) \rangle p_2(t - x') - \epsilon \langle \zeta(\xi) \rangle^2$$

where

$$\zeta(\xi) = \xi + ib|\xi|(x - x'), \; z = x + iy.$$

Then

$$\begin{aligned} &\Re Q(z, x', t, \xi, \epsilon) \\ &= -b|\xi||x - x'|^2 - y \cdot \xi - \Re \langle \zeta(\xi) \rangle^{\frac{l}{k}} p_1(t - x') - \Re \langle \zeta(\xi) \rangle p_2(t - x') \\ &- \epsilon \Re \langle \zeta(\xi) \rangle^2 \end{aligned}$$

We note that

$$\langle \zeta(\xi) \rangle^2 = \sum_{j=1}^{m} (\xi_j + ib|\xi|(x_j - x'_j))^2 = |\xi|^2 - b^2|\xi|^2|x - x'|^2 + i2b|\xi|\xi \cdot (x - x').$$

Let $|x| \leq 1$. Then since $|x'| \leq r$,

$$b^2|\xi|^2|x - x'|^2 \leq b^2 B|\xi|^2$$

for some $B > 0$. Then we can choose $b > 0$ small enough such that

$$\Re \langle \zeta(\xi) \rangle^2 = |\xi|^2 - b^2|\xi|^2|x - x'|^2 \geq \frac{|\xi|^2}{2}$$

and

$$\arg \langle \zeta(\xi) \rangle^2 \in \left[\frac{-\pi}{4}, \frac{\pi}{4} \right].$$

Hence

$$\begin{aligned} \Re \langle \zeta(\xi) \rangle^{\frac{l}{k}} &= \Re \left(\sum_{j=1}^{m} \zeta_j^2(\xi) \right)^{\frac{l}{2k}} = \Re \left(\langle \zeta(\xi) \rangle^2 \right)^{\frac{l}{2k}} \\ &= \Re e^{\frac{l}{2k} \log\left(\langle \zeta(\xi) \rangle^2 \right)} \\ &= \left| \langle \zeta(\xi) \rangle^2 \right|^{\frac{l}{2k}} \cos\left(\frac{l}{2k} \arg \langle \zeta(\xi) \rangle^2 \right) > 0, \end{aligned}$$

and

$$\begin{aligned}\Re\langle\zeta(\xi)\rangle &= \left|\langle\zeta(\xi)\rangle^2\right|^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\ &\geq \left(\Re\langle\zeta(\xi)\rangle^2\right)^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\ &= B'|\xi|,\ \ B' > 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Re Q(z, x', t, \xi, \epsilon)\\ &= -b|\xi||x-x'|^2 - y\cdot\xi - \Re\langle\zeta(\xi)\rangle^{\frac{l}{k}}p_1(t-x') - \Re\langle\zeta(\xi)\rangle p_2(t-x')\\ &- \epsilon\Re\langle\zeta(\xi)\rangle^2\\ &\leq -b|\xi||x-x'|^2 + |y||\xi| - B'c_3|\xi||t-x'|^{2k}\end{aligned}$$

Let $z = x + iy = 0$. Then

$$\Re Q(0, x', t, \xi, \epsilon) \leq -b|\xi||x'|^2 - B'c_3|\xi||t-x'|^{2k}.$$

If $|x'| \geq \frac{a}{2}$, then

$$\Re Q(0, x', t, \xi, \epsilon) \leq -b|\xi||x'|^2 \leq -b\frac{a^2}{4}|\xi|.$$

If $|x'| \leq \frac{a}{2}$, then since $|t| \geq a$, $|t-x'| \geq \frac{a}{2}$ and so

$$\Re Q(0, x', t, \xi, \epsilon) \leq -B'c_3|\xi||t-x'|^{2k} \leq -\frac{B'c_3a^{2k}}{2^{2k}}|\xi|.$$

Thus there is $A_1 > 0$ independent of $\epsilon > 0$ such that

$$\Re Q(0, x', t, \xi, \epsilon) \leq -A_1|\xi|,\ \ \forall|\xi| \geq 1.$$

By continuity and homogeneity in ξ, there is $\delta_3 > 0$ such that for some $A_2 > 0$

$$\Re Q(z, x', t, \xi, \epsilon) \leq -A_2|\xi|, \forall|\xi| \geq 1, |z| \leq \delta_3.$$

Therefore,

$$\left|\int_{|\xi|\geq 1}\omega(z, x', t, \zeta(\xi), \epsilon)d\xi\right| \leq C'\int_{|\xi|\geq 1}e^{-A_2|\xi|}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right|d\xi,$$

and so

$$|I_3^{\epsilon}(z)| \leq\leq A_3$$

for some $A_3 > 0$ independent of $\epsilon > 0$ for all $|z| < \delta_3$.

Let $\delta = \min\{1, \delta_2, \delta_3\}$. Then there is $0 < \lambda < \infty$ such that

$$\sup_{0<\epsilon\leq 1} |u_1^{\epsilon}(z)| \leq \lambda, \ \forall |z| < \delta.$$

Thus there is a subsequence $\epsilon_k > 0$ such that for some $0 < \delta' < \delta$,

$$u_1^{\epsilon_k}(x + iy) \to u_1(x + iy)$$

uniformly on $|x + iy| \leq \delta'$. In particular, $u_1(z)$ is holomorphic on $|z| < \delta$. Hence $(0, \xi^0) \notin WF_a(u_1)$ and so $(0, \xi^0) \notin WF_s(u_1)$. Since $WF_s(u) \subset WF_s(u_0) \cup WF_s(u_1)$ we get $(0, \xi^0) \notin WF_s(u)$ and the proof is complete.

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