A Microlocal Version of Bochner's Tube Theorem

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Introduction. The classical Bochner's theorem states that any holomorphic function in a tube $\mathbf{R}^n + i\Omega \subset \mathbf{C}^n$, with Ω a domain in $\mathbf{R}^n$, extends holomorphically to $\mathbf{R}^n + i \operatorname{ch} \Omega$ (ch : convex hull).

This result has been generalized to "tuboïds" $\Omega' + i\Omega$ (with Ω' also a domain in $\mathbf{R}^n$) by H. Komatsu [9] (of course, in this case one cannot extend the holomorphic functions to convex hulls). A different generalization, in which $\Omega + i\mathbf{R}^n$ is replaced by $M + i\mathbf{R}^n$, with M a submanifold of $\mathbf{R}^n$, has been obtained by Kazlow [8]. This, needless to say, is a very small portion of the literature of the so-called CR functions. Noteworthy in this area are the paper of H. Lewy [10] and the book of Hörmander [6] (see also Hill [3], Hunt-Wells [7], etc.).

The present article contains a microlocal version of Bochner's theorem. Although we prefer to reason on the analytic wave-front set of the CR-functions, our results can be rephrased in terms of holomorphic extensions inside "short" cones with vertices at points of the "tuboïd manifold." In the tube case the geometric equivalent of analytic hypoellipticity (see below) has been announced in Hill-Kazlow [4].

Actually we analyze a somewhat more general situation: we study the functions (or distributions) annihilated by systems of pairwise commuting vector fields whose coefficients reflect the tube aspect of the situation. In this setup we completely characterize the analytic wave-front set of those solutions. Among other things this yields necessary and sufficient conditions for analytic hypoellipticity of the systems under consideration.

I. The basic result and its corollaries. We shall denote by $t = (t_1,\ldots,t_m)$ the variable in $\mathbf{R}^m$ and by $x = (x_1,\ldots,x_n)$ the variable in $\mathbf{R}^n$. Let U denote an open and connected subset of $\mathbf{R}^m$, ϕ a Lipschitz continuous mapping $U \to \mathbf{R}^n$. We shall use the notation

$$z = x + i\,\phi(t), \qquad i = \sqrt{-1}, \tag{1.1}$$

and $z = (z_1,\ldots,z_n) \in \mathbf{C}^n$, $z_k = x_k + i\,\phi_k(t)$. We consider the associated complex vector fields in $U \times \mathbf{R}^n$:

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{n} \frac{\partial \phi_k}{\partial t_j}(t) \frac{\partial}{\partial x_k}, \; j = 1, \ldots, m. \tag{1.2}$$

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We have

(1.3) $$L_j z_k = 0 \qquad j = 1, \ldots, m, \; k = 1, \ldots, n.$$

Let V be an open subset of $\mathbf{R}^n$. We shall write

(1.4) $$\Omega = U \times V.$$

If h is a Lipschitz continuous solution in Ω of the system of equations

(1.5) $$L_j h = 0, \qquad j = 1, \ldots, m,$$

and $t^0 \in U$, we are interested in the analytic wave-front set of the function $x \mapsto h(t^0,x)$. More precisely if $x^0 \in V$ and $\xi^0 \in \mathbf{R}^n \backslash 0$, we shall give a sufficient condition insuring that (x^0,ξ^0) is not in the analytic wave-front set of $h(t^0,x)$.

It is convenient to assume in the sequel that the central point (t^0,x^0) is the origin of $\mathbf{R}^m \times \mathbf{R}^n$ and that V is an open ball centered at the origin of $\mathbf{R}^n$ of radius $r > 0$. If h is a Lipschitz solution in Ω of (1.5), we shall write

(1.6) $$h_0(x) = h(0,x).$$

We also assume that $\phi(0) = 0$.

Theorem 1.1. *Let $\xi^0 \in \mathbf{R}^n \backslash 0$ and assume there are $t^* \in U \backslash 0$ and a Lipschitz curve γ in U with 0 and t^* as its endpoints satisfying:*

(1.7) $$-\phi(t^*) \cdot \xi^0 > 0,$$

(1.8) $$\sup_{t \in \gamma} |\phi(t)| < r,$$

(1.9) $$|\phi(t^*)|^2 \sup_{t \in \gamma} \phi(t) \cdot \xi^0 < [r^2 - \sup_{t \in \gamma} |\phi(t)|^2][-\phi(t^*) \cdot \xi^0]$$

Then if h is any Lipschitz continuous solution of (1.5) in Ω, $(0,\xi^0)$ is not in the analytic wave-front set of h_0 defined by (1.6).

Proof. For the sake of simplicity we shall assume $|\xi^0| = 1$. Let ε, $0 < \varepsilon < 1$, and $K > 0$ to be determined later. Let $g \in C_0^\infty(V)$, $g(x) \equiv 1$ for $|x| \leq (1 - \varepsilon)r$. If h is a Lipschitz solution of (1.5) in Ω and $(x,\xi) \in \mathbf{R}^{2n}$, consider the integral

(1.10) $$I(x,\xi) = \int_{\mathbf{R}^n} \int_\gamma e^{i(x-y-i\phi(t))\cdot\xi - K(x-y-i\phi(t))^2|\xi|} L[g(y)h(t,y)]dtdy.$$

We have used the notation $z^2 = \sum_{j=1}^n z_j^2$, and

$$Lf(t,y)dt = \sum_{j=1}^m L_j f(t,y)\, dt_j$$

which is a one-form on U depending on y.

Integrating (1.10) by parts with respect to t and y and using (1.3) we obtain

(1.11) $$I(x,\xi) = I_*(x,\xi) - I_0(x,\xi)$$

with

$$I_*(x,\xi) = \int_{\mathbf{R}^n} e^{i(x-y-i\phi(t^*))\cdot\xi - K(x-y-i\phi(t^*))^2|\xi|}\, g(y)h(t^*,y)dy \tag{1.12}$$

$$I_0(x,\xi) = \int_{\mathbf{R}^n} e^{i(x-y)\xi - K(x-y)^2|\xi|}\, g(y)h_0(y)dy. \tag{1.13}$$

In order to show that $(0,\xi^0)$ is not in the analytic wave-front set of h_0 it suffices to show that the estimate

$$|I_0(x,\xi)| \le C e^{-|\xi|/c}, \tag{1.14}$$

with $C > 0$, holds for (x,ξ) in a conic neighborhood of $(0,\xi^0)$ (see Sjöstrand [11]). In view of (1.11) we shall actually prove similar estimates for $I(x,\xi)$ and $I_*(x,\xi)$.

First note that we have

$$\left|e^{i(x-y-i\phi(t))\cdot\xi - K(x-y-i\phi(t))^2|\xi|}\right| = e^{-E(t,x,y,\xi)|\xi|} \tag{1.15}$$

with

$$E(t,x,y,\xi) = -\phi(t)\frac{\xi}{|\xi|} + K[|x-y|^2 - |\phi(t)|^2]. \tag{1.16}$$

On the other hand if $y \in \operatorname{supp} g$, we have

$$\begin{aligned} E(t,x,y,\xi) &\ge -\phi(t)\cdot\xi^0 + K(|y|^2 - |\phi(t)|^2) \\ &\quad - |\phi(t)|\left|\frac{\xi}{|\xi|} - \xi^0\right| - 2Kr|x|. \end{aligned} \tag{1.17}$$

If $r_0 > 0$, $\rho_0 > 0$ we write

$$\Gamma(r_0,\rho_0) = \left\{(x,\xi) : |x| < r_0, \left|\frac{\xi}{|\xi|} - \xi^0\right| < \rho_0\right\}.$$

Making use of (1.8) we have for $(x,\xi) \in \Gamma(r_0,\rho_0)$ and $t \in \gamma$,

$$E(t,x,y,\xi) \ge -\phi(t)\cdot\xi^0 + K(|y|^2 - |\phi(t)|^2) - r\rho_0 - 2Krr_0. \tag{1.18}$$

Estimate of $I_*(x,\xi)$. It follows from (1.18) that we have

$$E(t^*,x,y,\xi) \ge -\phi(t^*)\cdot\xi^0 - K|\phi(t^*)|^2 - r(\rho_0 + 2Kr_0). \tag{1.19}$$

Therefore if

$$0 < K < \frac{-\phi(t^*)\cdot\xi^0}{|\phi(t^*)|^2} \tag{1.20}$$

and r_0, ρ_0 are small enough we get

$$E(t^*,x,y,\xi) \geq c > 0$$

for $(x,\xi) \in \Gamma(r_0,\rho_0)$ and $y \in \operatorname{supp} g$, which proves an estimate of the type (1.14) for $I_*(x,\xi)$.

Estimate of $I(x,\xi)$. Since h satisfies (1.5) and $g(y) \equiv 1$ for $|y| \leq (1-\varepsilon)r$, in integral (1.10) we have $(1-\varepsilon)r \leq |y| \leq r$, and from (1.18) we get

$$E(t,x,y,\xi) \geq -\phi(t)\cdot\xi^0 + K[(1-\varepsilon)^2r^2 - |\phi(t)|^2] - r(\rho_0 + 2Kr_0). \tag{1.21}$$

Now using (1.8) and (1.9) we can choose ε, $0 < \varepsilon < 1$, satisfying

$$\sup_{t\in\gamma}|\phi(t)| < (1-\varepsilon)r,$$

$$|\phi(t^*)|^2 \sup_{t\in\gamma}\phi(t)\cdot\xi^0 < [r^2(1-\varepsilon)^2 - \sup_{t\in\gamma}|\phi(t)|^2][-\phi(t^*)\cdot\xi^0],$$

and then $K > 0$ satisfying

$$\frac{\sup_{t\in\gamma}\phi(t)\cdot\xi^0}{r^2(1-\varepsilon)^2 - \sup_{t\in\gamma}|\phi(t)|^2} < K < -\frac{\phi(t^*)\cdot\xi^0}{|\phi(t^*)|^2}. \tag{1.22}$$

Note that (1.22) implies (1.20). Again choosing r_0 and ρ_0 small enough we obtain from (1.21) and (1.22) that for $t \in \gamma$, $(x,\xi) \in \Gamma(r_0,\rho_0)$ and $(1-\varepsilon)r \leq |y| \leq r$,

$$E(t,x,y,\xi) \geq c > 0,$$

which yields an estimate of the type (1.14) for $I(x,\xi)$. Q.E.D.

We give some corollaries of Theorem 1.1.

Corollary 1.1. *Let $\xi^0 \in \mathbf{R}_n\backslash 0$ and assume there is a Lipschitz curve γ_0 in U with 0 as its endpoint satisfying*

$$\phi(t)\cdot\xi^0 < 0 \qquad \textit{for all } t \in \gamma_0\backslash 0. \tag{1.23}$$

Then for any Lipschitz continuous solution h of (1.5) defined in some neighborhood of the origin of $\mathbf{R}^m \times \mathbf{R}^n$, $(0,\xi^0)$ is not in the analytic wave-front set of h_0.

Proof. Possibly by shrinking U we can assume that h is defined in $U \times V$, where V is a ball centered at the origin of radius $r > 0$.

We claim that the assumptions of Theorem 1.1 are satisfied. Indeed we can choose $t^* \in U\backslash 0$ close enough to 0 so that if γ is the portion of γ_0 joining 0 and t^* we have (1.8). Condition (1.7) follows from (1.23). Condition (1.9) is valid since $\sup_{t\in\gamma}\phi(t)\cdot\xi^0 = 0$ (which also follows from (1.23) and the fact that $\phi(0) = 0$). Corollary 1.1 is then a consequence of Theorem 1.1. Q.E.D.

Corollary 1.2. *Let $\xi^0 \in \mathbf{R}^n \backslash 0$. Assume there is $t^* \in U$ such that (1.7) holds. Then if h is a Lipschitz continuous solution of (1.5) in $U \times \mathbf{R}^n$ then $(0,\xi^0)$ is not in the wave-front set of h_0.*

Proof. Let γ be any Lipschitz curve connecting 0 and t^*. Choosing r large enough so that (1.8) and (1.9) hold, Theorem 1.1 can be used. Q.E.D.

If (a_ν) and (b_ν) are two sequences of real numbers, $b_\nu > 0$, we write

$$a_\nu << b_\nu$$

if

$$\lim_{\nu\to\infty}\left[\max\left(0, \frac{a_\nu}{b_\nu}\right)\right] = 0.$$

Corollary 1.3. *Let $\xi^0 \in \mathbf{R}^n \backslash 0$. Assume there is a sequence $t_\nu \in U \backslash 0$, $\lim_{\nu\to\infty} t_\nu = 0$ and a sequence of Lipschitz curves γ_ν with endpoints 0 and t_ν satisfying*

$$\phi(t_\nu)\cdot\xi^0 < 0, \tag{1.24}$$

$$\lim_{\nu\to\infty} [\sup_{t\in\gamma_\nu} |\phi(t)|] = 0, \tag{1.25}$$

$$|\phi(t_\nu)|^2 \sup \phi(t)\cdot\xi^0 << -\phi(t_\nu)\cdot\xi^0. \tag{1.26}$$

Then the conclusion of Corollary 1.1 holds.

Proof. Possibly by contracting U we can assume that h is defined in $U \times V$ where V is a ball centered at the origin of $\mathbf{R}^n$ with radius $r > 0$. In view of (1.25) there is $\nu_0 \in \mathbf{Z}_+$ such that if $\nu \geq \nu_0$, $t_\nu \in U$ and

$$\sup_{t\in\gamma_\nu} |\phi(t)| < \frac{r}{\sqrt{2}}. \tag{1.27}$$

Making use of (1.26) there is $\nu_1 \geq \nu_0$ such that

$$|\phi(t_{\nu_1})|^2 \sup_{\gamma_{\nu_1}} [\phi(t)\cdot\xi^0] < \frac{r^2}{2}[-\phi(t_{\nu_1})\cdot\xi^0]. \tag{1.28}$$

Choosing $t^* = t_{\nu_1}$, $\gamma = \gamma_{\nu_1}$, (1.7), (1.8) and (1.9) follow immediately from (1.24), (1.27) and (1.28), therefore Theorem 1.1 can be applied Q.E.D.

Remark 1.1. The sign conditions of the type (1.7), (1.23) are essential in the previous results. If $\phi(t)\cdot\xi^0 \geq 0$ for all $t \in U$ then there is a Lipschitz function h satisfying (1.5) such that $(0,\xi^0)$ is in the analytic wave-front set of h_0. Indeed take

$$h(t,x) = (x\cdot\xi^0 + i\phi(t)\cdot\xi^0)^{3/2}$$

with the principal determination of $\zeta^{3/2}$ for $\zeta \in \mathbf{C}, I_m\zeta \geq 0$. It is easy to check that $(0,\xi^0)$ belongs to the analytic wave-front set of $h_0(x) = (x \cdot \xi^0)^{3/2}$.

If ϕ is a *real-analytic* function, we can give a complete characterization of the analytic wave front set of h_0. We have

Corollary 1.4. *Let $\xi^0 \in \mathbf{R}^n\backslash 0$ and assume ϕ to be real analytic in U. Then the following conditions are equivalent:*

i) The origin of $\mathbf{R}^n$ is not a local minimum of the function $t \mapsto \phi(t) \cdot \xi^0$.

ii) For every distribution h defined in some neighborhood of 0 in $\mathbf{R}^{n+m}$ and satisfying (1.5), $(0,\xi^0)$ is not in the analytic wave-front set of h_0.

Proof. The fact that ii) implies i) follows from Remark 1.1. It remains to show that i) implies ii). Assume that i) holds. Consider the set

$$S = \{t \in U : \phi(t)\xi^0 < 0\}.$$

Since $\phi(0) = 0$ and the origin is not a local minimum of $\phi(t) \cdot \xi^0$ we conclude that S is not empty and that the origin is in the closure of S. Because S is a subanalytic set, we can find an analytic curve γ_0 starting from the origin such that $\{\gamma_0\backslash 0\} \subset S$ (see for example Hironaka [5]). If h is a C^1 solution of (1.5) defined in a neighborhood of the origin, the conclusion (i.e. condition ii)) follows from Corollary 1.1. If h is a distribution solution of (1.5) in some neighborhood of the origin Ω', it is shown in [1] that, possibly after shrinking Ω', we can write

$$h = \Delta_x^q f \qquad \text{where} \quad \Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

$q \in \mathbf{Z}_+$ and $f \in C^1(\Omega')$ satisfying (1.5). From the preceding step we conclude that $(0,\xi^0)$ is not in the analytic wave-front set of $f(0,x)$; since $h_0(x) = \Delta_x^q f(0,x)$, the sought conclusion (i.e. condition ii)) immediately follows. Q.E.D.

II. Application to analytic hypoellipticity. We assume in this section that ϕ is real analytic in an open connected set U of $\mathbf{R}^m$. We denote by $\mathbf{L}$ the family of vector fields $L_1, \ldots, L_m$ defined by (1.2) in $U \times \mathbf{R}^n$. As in [1] we introduce the following definition.

Definition 2.1. We say that $\mathbf{L}$ is *analytic hypoelliptic at* $(t^0,x^0) \in U \times \mathbf{R}^n$ if any distribution u in some open neighborhood ω of (t^0,x^0), such that $L_j u$ is analytic in ω, for $j = 1, \ldots, m$, is itself analytic in a possibly smaller open neighborhood ω' of (t^0,x^0).

We say that $\mathbf{L}$ is *analytic hypoelliptic in a subset of* $U \times \mathbf{R}^n$ if $\mathbf{L}$ is analytic hypoelliptic at each point of that subset. Since the coefficients of L_j are independent of x, it is clear that $\mathbf{L}$ is analytic hypoelliptic at $(t^0,x^0) \in U \times \mathbf{R}^n$ if and only if $\mathbf{L}$ is analytic hypoelliptic in $\{t_0\} \times \mathbf{R}^n$. We have:

Proposition 2.1. *The system* **L** *is analytic hypoelliptic at* $(t^0,x^0) \in U \times \mathbf{R}^n$ *if and only if for every distribution h defined in some neighborhood of* (t^0,x^0) *and satisfying (1.5), the distribution* $h(t^0,\cdot)$ *is analytic in some neighborhood of* x^0.

Proof. It is clear that the condition given above is necessary for the analytic hypoellipticity of **L** at (t^0,x^0). Let us prove that it is sufficient. Assume that u is a distribution in an open neighborhood of (t^0,x^0) such that $L_ju = f_j$ is analytic near (t^0,x^0) for $j = 1, \ldots, m$. We can find an analytic function v near (t^0,x^0) such that

$$L_jv = f_j \qquad j = 1, \ldots, m;$$

therefore, $L_j(u - v) = 0$ for $j = 1, \ldots, m$. Assuming that the condition stated in the proposition holds, we conclude that $u(t^0,\cdot)$ is analytic in some neighborhood of x^0. By a standard argument (Cauchy-Kovalevky and Holmgren type theorems easily derived for the system **L**) we conclude that u itself is analytic in some neighborhood of (t^0,x^0). Q.E.D.

The following result is an immediate consequence of Corollary 1.4 and Proposition 2.1.

Theorem 2.1. *The system* **L** *is analytic hypoelliptic at* $(t^0,x^0) \in U \times \mathbf{R}^n$ *if and only if for every* $\xi \in \mathbf{R}^n\backslash 0$, t^0 *is not a local extremum of the function* $t \mapsto \phi(t)\cdot\xi$.

Example 2.1. ***Maire's example.*** Take $m = n = 2$,

$$\phi_1(t) = -3t_1,$$

$$\phi_2(t) = (t_1t_2 + 1)\, t_1^3.$$

It is shown in [1] (by different methods) that the corresponding system (L_1,L_2) is analytic hypoelliptic in $\mathbf{R}^4$. Making use of Theorem 2.1 such a result is now straightforward.

Case of a single vector field. Take $m = 1$. Let ϕ be an analytic function in an open set U of $\mathbf{R}$, valued in $\mathbf{R}^n$ and

$$L = \frac{\partial}{\partial t} - i \sum_{k=1}^{n} \frac{\partial \phi_k}{\partial t_k}(t) \frac{\partial}{\partial x_k}. \tag{2.1}$$

Let $t^0 \in U$, the Taylor expansion of ϕ at t^0 can be written

$$\phi(t) - \phi(t^0) = \sum_{p=1}^{\infty} (t - t^0)^p\, v_p. \tag{2.2}$$

with $v_p \in \mathbf{R}^n$. We have:

Corollary 2.1. *Let* $t^0 \in U$ *and* $x^0 \in \mathbf{R}^n$, *the vector field L is analytic hypoelliptic at* (t^0,x^0) *if and only if the vectors* v_p*'s given in (2.2) satisfy:*

i) The v_p*'s span all of* $\mathbf{R}^n$.
ii) If p *is even then* v_p *is a linear combination of* v_k, $1 \le k \le p - 1$.

Proof. Making use of Theorem 2.1 we must show that i) and ii) are equivalent to the fact that for every $\xi \in \mathbf{R}^n \backslash 0$, t^0 is not a local extremum of $\phi(t) \cdot \xi$. Assume i) and ii) hold and let $\xi \in \mathbf{R}^n \backslash 0$. We have

$$\phi(t) \cdot \xi - \phi(t^0) \cdot \xi = \sum_{p=1}^{\infty} (t - t^0)^p v_p \cdot \xi. \tag{2.3}$$

There is at least one p such that $v_p \cdot \xi \neq 0$, and the first such p is necessarily odd, which proves that t_0 is not a local extremum of $\phi(t) \cdot \xi$. Conversely, if t_0 is not a local extremum, then the series (2.3) must start with a nonzero odd term for every choice of $\xi \in \mathbf{R}^n \backslash 0$, which easily implies i) and ii). Q.E.D.

Example 2.2. Let $p_k \in \mathbf{Z}_+$ for $k = 1, \ldots, n$. The vector field

$$\frac{\partial}{\partial t} + i \sum_{i=1}^{n} t^{p_k} \frac{\partial}{\partial x_k}$$

is analytic hypoelliptic at the origin of $\mathbf{R}^{n+1}$ if and only if the p_k's are even and distinct.

III. Extendability of CR functions. Let U be an open set of $\mathbf{R}^m$ and ϕ a Lipschitz continuous mapping $U \to \mathbf{R}^n$. Let V be an open set of $\mathbf{R}^n$. We shall denote by $z(U \times V)$ the image of $U \times V$ under the mapping z defined by (1.1), regarded as a subset of $\mathbf{C}^n$.

Definition 3.1. A function u defined on the set $z(U \times V)$ is said to be *Lipschitz continuous,* if its pull-back via z, $\tilde{u} = u \circ z$, is Lipschitz continuous on $U \times V$. Moreover u is said to *satisfy the induced Cauchy-Riemann equations,* or to be a CR *function,* if $\tilde{u}$ satisfies Equation (1.5) in $U \times V$.

Some justification for this definition is needed. We observe that the push via z of L_j, $1 \le j \le m$, regarded as a complex vector field tangent to $U \times V$ at (t,x) is equal to

$$\sum_{k=1}^{n} (L_j z_k) \frac{\partial}{\partial z_k} + (L_j \bar{z}_k) \frac{\partial}{\partial \bar{z}_k} = -2i \sum_{k=1}^{n} \frac{\partial \phi_k(t)}{\partial t_j} \frac{\partial}{\partial \bar{z}_k} \tag{3.1}$$

where we have used (1.3). If ϕ is, say of class C^1, and the image of U is an immersed submanifold of $\mathbf{R}^n$, the vector fields (3.1) span the entire Cauchy-Riemann operator tangent to $z(U \times V)$ as (t,x) varies in $U \times V$. In this case u is a CR function according to our definition if and only if u satisfies the usual induced Cauchy-Riemann equations, i.e. u is annihilated by all complex vector fields tangent to $z(U \times V)$ which are linear combinations of $\partial/\partial \bar{z}_k$, $k = 1, \ldots, n$.

If f is a holomorphic function in an open neighborhood of $z(U \times V)$ in $\mathbf{C}^n$, clearly the restriction of f to $z(U \times V)$ is a CR function, in the sense of Definition

3.1. We shall investigate the following local extendability question. Let $(t^0,x^0) \in U \times V$ and u be a CR function defined on $z(U \times V)$, when does u extend holomorphically to a neighborhood of $z(t^0,x^0)$? We have the following result:

Proposition 3.1 *Let u be a* CR *function defined on $z(U \times V)$ and $(t^0,x^0) \in U \times V$. The function u extends holomorphically to a neighborhood of $z(t^0,x^0)$ if and only if the function*

$$x \to \tilde{u}(t^0,x) = u(z(t^0,x))$$

is (real) analytic at x^0.

Proof. Clearly the condition stated in Proposition 3.1 is necessary. Let us show that it is sufficient. Assume that $\tilde{u}(t^0,x)$ is analytic in some neighborhood of x^0. Set $h_0(x) = \tilde{u}(t^0,x)$. The functions $\tilde{u}(t,x)$ and $h_0(x + i[\phi(t) - \phi(t_0)])$ are both solutions of (1.5) in some neighborhood of (t^0,x^0) in $U \times V$, and they are equal for $t = t_0$, by uniqueness in the Cauchy problem for the system (1.5) (see footnote at end of paper). We obtain that

$$\tilde{u}(t,x) = u(z(t,x)) = h_0(x + i[\phi(t) - \phi(t_0)]) \tag{3.2}$$

near (t^0,x^0).

We easily conclude from (3.2) that, in some neighborhood of $z(t^0,x^0)$ in $\mathbf{C}^n$, u is the restriction to $z(U \times V)$ of the holomorphic function

$$z \mapsto h_0(z - i\phi(t^0))$$

defined for z near $z(t^0,x^0)$ in $\mathbf{C}^n$. Q.E.D.

Making use of Proposition 3.1, it is clear that each result of Sections I and II can yield a local extendability result for CR functions defined on $z(U \times V)$. In fact, according to the general result in [1, Theorem 2.2], locally all solutions of (1.5) are of the form $h \circ z$ where h is a CR function. To avoid several repetitions, we restrict ourselves to restating Theorem 2.1 in this context.

Theorem 3.1. *Assume ϕ to be analytic in U and let $(t^0,x^0) \in U \times V$. Any* CR *function defined on $z(U \times V)$ extends holomorphically to a neighborhood of $z(t^0,x^0)$ if and only if for every $\xi \in \mathbf{R}^n\backslash 0$, t^0 is not a local extremum of the function $t \to \phi(t) \cdot \xi$.*

In fact the microlocal results of Sections I and II can yield holomorphic extendability of CR functions not only to a full neighborhood of a point in $z(U \times V)$ in $\mathbf{C}^n$, but also to open sets of $\mathbf{C}^n$ whose boundary contains part of $z(U \times V)$. This observation is based on the description of the analytic wave-front set of the solution of (1.5) given in the previous sections and the following lemma.

If Γ is a convex cone of $\mathbf{R}^n$ we set

$$\Gamma^0 = \{y \in \mathbf{R}^n : y \cdot \xi \geq 0, \forall\, \xi \in \Gamma\},$$

and if $\delta > 0$

$$\Gamma_\delta^0 = \{y \in \Gamma^0, |y| < \delta\}.$$

Lemma 3.1. *Let Γ be a closed convex cone of $\mathbf{R}^n\backslash\{0\}$ and u a continuous function defined in an open set ω of $\mathbf{R}^n$, whose analytic wave-front is contained in $\omega \times \Gamma$. Then for every ω' relatively compact open subset of ω and every open convex cone Γ' containing Γ, there is $\delta > 0$ such that u extends holomorphically to $\omega' + \sqrt{-1}\,\mathrm{int}\,(\Gamma_\delta'^0)$.*

Proof. Assume ω' and Γ' to be as given in the lemma. By multiplying u with a cut-off function we may assume that u has compact support in $\mathbf{R}^n$. Let Γ'' be an open convex cone of $\mathbf{R}^n$ containing Γ and whose closure is contained in Γ'. Making use of Lemma 1.6, Chapter 5, of Treves [12] there is a C^∞ function g in $\mathbf{R}^n\backslash 0$ such that $g(\xi) = 1$ for $\xi \in \Gamma''$, $g(\xi) = 0$ outside Γ' and

$$\int e^{ix\xi}\,(1 - g(\xi))\,\hat{u}(\xi)\,d\xi$$

is analytic in an open neighborhood of $\bar{\omega}'$ in ω. For $x \in \omega'$ we have

$$u(x) = \frac{1}{(2\pi)^n}\int e^{ix\xi}\,g(\xi)\,\hat{u}(\xi)\,d\xi + \frac{1}{(2\pi)^n}\int e^{ix\xi}\,(1 - g(\xi))\hat{u}(\xi)\,d\xi. \tag{3.3}$$

Since $g(\xi) = 0$ outside Γ', the first integral can be holomorphically extended to $\mathbf{R}^n + \sqrt{-1}\,\mathrm{int}(\Gamma'^0)$. Therefore the conclusion of the lemma follows from the analyticity of the second integral in a neighborhood of $\bar{\omega}'$.

Remark 3.1. It should be mentioned that other extendability results generalizing Bochner's tube theorem appear in the literature: Hörmander [6], Komatsu [9], Kazlow [8]. The basic ingredient in these papers is the so-called folding screen lemma. A local version of such a lemma can easily be proved using Lemma 3.1 and Theorem 1.1 in this paper.

Remark 3.2. An almost converse of Lemma 3.1 holds: If u is continuous in ω and extends holomorphically to $\omega + \sqrt{-1}\,(\mathrm{int}\,\Gamma_\delta^0)$, ($\Gamma$ closed and convex cone, $\delta > 0$) then the analytic wave-front set of u is contained in $\omega \times \Gamma$. This can be proved, for example, by using Sjöstrand's definition of the analytic wave-front set and a deformation of the contour of integration. It should be noted that the basic ideas of Lemma 3.1 and this remark can be found in Bony [2].

Footnote. We have used the following uniqueness result: If h is a Lipschitz continuous function satisfying (1.5) in $U \times V$ and if for $(t^0,x^0) \in U \times V$, $h(t^0,x) \equiv 0$ for x in some neighborhood of x^0 in V, then $h(t,x) \equiv 0$ for (t,x) in some neighborhood of (t^0,x^0) in $U \times V$. If ϕ were real analytic this result would essentially be Holmgren's theorem. Here ϕ is assumed to be only Lipschitz continuous. This uniqueness result follows from Remark 2.2 in [1] (the fact that the coefficients of L_j's are only L^∞ or equivalently that z is Lipschitz in t, does not affect the conclusion nor the proof in [1]).

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