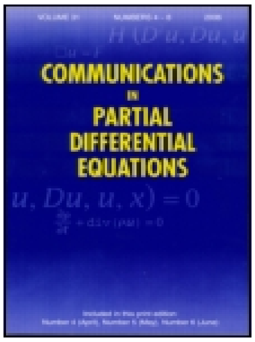

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A Class of FBI Transforms

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We introduce a class of FBI transforms whose phase functions may have a degenerate Hessian and present an application of these transforms to the microlocal analytic hypoellipticity of certain systems of vector fields.

Keywords FBI transform; Microlocal analyticity; Microlocal smoothness.

Mathematics Subject Classification Primary 32A07; Secondary 32D10, 32D15.

1. Introduction

This paper introduces a class of FBI transforms that can be used to characterize the microlocal smoothness and microlocal analyticity of functions. The classical and more commonly used FBI transform has the form

$$\mathfrak{F}u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^2} u(x')dx', \quad x, \xi \in \mathbb{R}^m \tag{1.1}$$

where u is a continuous function of compact support in $\mathbb{R}^m$ or a distribution of compact support in which case the integral is understood in the duality sense. This transform characterizes microlocal analyticity (see [14, 18]) and microlocal smoothness (see [10]) and has been used in numerous works to study the regularity of solutions of linear and nonlinear partial differential equations.

A more general version of (1.1) has also been used extensively in studying the regularity of solutions of overdetermined systems of locally integrable complex vector fields, in particular, in the study of the holomorphic extendability of CR functions. To describe this more general version, suppose $L_1, \ldots, L_n$ is a system of linearly independent, smooth complex vector fields in $\mathbb{R}^{m+n}$ where we use the variables (x, y) for a point in $\mathbb{R}^{m+n}$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Assume the functions $Z_k(x, y) = x_k + i\varphi_k(x, y)$, $k = 1, \ldots, m$ are first integrals for the L_j, that is, $L_jZ_k = 0$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$ and the φ_j are real-valued smooth functions. If $u(x, y)$ is a continuous function or a distribution and $g(x)$ is a smooth function of

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compact support with $g(x) \equiv 1$ near $0 \in \mathbb{R}^m$, then the more general FBI transform has the form (see [1, 19]):

$$\mathfrak{F}u(z, \zeta) = \int_{\mathbb{R}^m} e^{i\zeta\cdot(z-Z(x',0))-\langle\zeta\rangle(z-Z(x',0)^2} g(x')u(x', 0)dZ, \tag{1.2}$$

where $z, \zeta \in \mathbb{C}^m$ and $dZ = dZ_1 \wedge \ldots dZ_m$. We have used the notations $Z(x, 0) = (Z_1(x, 0), \ldots, Z_m(x, 0))$, and for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, $z^2 = \sum_{j=1}^m z_j^2$. For $\zeta \in \mathbb{C}^m$ near real space, $\langle\zeta\rangle$ denotes a branch of the square root of $\sum_{j=1}^m \zeta_j^2$.

Among the many works where the transforms (1.1) and (1.2) have been used, we mention [2–5, 10–13, 15, 18]. In [18] (see also [10, 20]) more general FBI transforms than (1.1) were considered where the phase function behaved much like the quadratic phase $i\xi \cdot (x - x') - |\xi||x - x'|^2$ in that the real part of the Hessian was required to be negative definite.

In this paper we introduce a class of transforms where the real part of the Hessian of the phase function may degenerate at the point of interest. We will demonstrate by means of a class of examples a situation where it is easier to apply our transforms than the more classical ones. Examples of the transforms we will introduce include, for each $k = 1, 2, \ldots,$

$$\mathfrak{F}_k u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^{2k}} u(x')dx', \quad x, \xi \in \mathbb{R}^m.$$

Observe that for $k > 1$, these transforms have a degenerate Hessian at the origin.

There are some instances where all global solutions of a system of vector fields on a fixed domain Ω may enjoy certain regularity (like analyticity) at a fixed point $p \in \Omega$ although locally defined solutions may not share that property. A classical example of this kind is provided by the Bochner extension theorem for tube domains in $\mathbb{C}^n$ and its various extensions to CR manifolds as in [7–9]. Another example is provided in the CR setting by a theorem independently proved by Joricke and Merker [16] according to which, if a generic CR submanifold $\mathcal{M}$ of $\mathbb{C}^n$ is globally minimal at a point $z \in \mathcal{M}$, then every continuous function u that is CR on all of $\mathcal{M}$ extends to a holomorphic function in a wedge at every point in the CR orbit of z. The application we present in Section 5 may also be viewed as an example that applies to global solutions.

There is another definition of microlocal analyticity of a distribution u in $\mathbb{R}^m$ in terms of expressing it as the boundary value of holomorphic functions of tempered growth defined on wedges in $\mathbb{C}^m$ (see [17]). The equivalence of this definition with the one using the FBI transform is shown in [18].

The referee pointed out that for the generalized FBI transform introduced in this work, it is possible to bring into play a semi-classical small parameter as it has been done in the case of the classical FBI, an approach that leads to alternative proofs of some of the results presented here thus avoiding a limiting process in the inversion formula presented in Section 2.

Section 2 introduces a class of FBI transforms for which an inversion formula is established. Section 3 contains a characterization of the C^∞ wave front using the transforms in Section 2. In Section 4 we characterize microlocal analyticity using a subclass of the transforms in Section 2. Section 5 is devoted to applications.

2. A General Inversion Formula

Consider a function $\psi \in \mathscr{S}(\mathbb{R}^m)$ satisfying $\int \psi(x)dx = 1$. For any number $\lambda > 0$, we define a "generalized FBI transform" (with generating function ψ and parameter λ) of a compactly supported continuous function $u(x) \in C_c^0(\mathbb{R}^m)$ by the formula

$$\mathfrak{F}_{\psi,\lambda}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')}\psi\big(|\xi|^\lambda(x-x')\big)u(x')dx', \quad x,\xi \in \mathbb{R}^m.$$

We also consider the FBI transform of a compactly supported distribution $u(x) \in \mathscr{E}'(\mathbb{R}^m)$ by interpreting the integral as the duality bracket between smooth functions and the distribution with compact support, i.e.,

$$\mathfrak{F}_{\psi,\lambda}u(x,\xi) = \left\langle u(x'), e^{i\xi\cdot(x-x')}\psi\big(|\xi|^\lambda(x-x')\big)\right\rangle, \quad x,\xi \in \mathbb{R}^m.$$

The map $u \mapsto \mathfrak{F}_{\psi,\lambda}u$ is always injective, in fact, there is an explicit inversion formula. Let $\chi \in \mathscr{S}(\mathbb{R}^m)$ satisfy $\int \chi(x)dx = 1$ and set

$$\sigma(\xi) = \frac{\widehat{\chi}(\xi)}{(2\pi)^m},$$

where $\widehat{\chi}$ denotes the Fourier transform of χ.

Lemma 2.1. *Let $u(x) \in \mathscr{E}'(\mathbb{R}^m)$ and set*

$$u_\varepsilon(x) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\varepsilon\xi)\mathfrak{F}_{\psi,\lambda}u(t,\xi)|\xi|^{\lambda m}\,dt\,d\xi. \tag{2.1}$$

Then $u_\varepsilon \to u$ in $\mathscr{E}'(\mathbb{R}^m)$ as $\varepsilon \searrow 0$. If $u(x) \in C_c^0(\mathbb{R}^m)$, $u_\varepsilon \to u$ uniformly.

Proof. Assume first that $u(x) \in C_c^0(\mathbb{R}^m)$. Since $\int \psi = 1$, a change of variables shows that

$$\int_{\mathbb{R}^m} \psi\big(|\xi|^\lambda(t-x')\big)|\xi|^{\lambda m}dt = 1, \quad x',\xi \in \mathbb{R}^m. \tag{2.2}$$

In the right hand side of (2.1), replace $\mathfrak{F}_{\psi,\lambda}u(t,\xi)$ by its defining formula

$$\int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')}\psi\big(|\xi|^\lambda(t-x')\big)u(x')dx'$$

so as to get a triple integral in (2.1). Since $e^{i\xi\cdot(x-t)}e^{i\xi\cdot(t-x')} = e^{i\xi\cdot(x-x')}$, performing the integration with respect to t first and taking account of (2.2) we obtain, in view of the inversion formula for the Fourier transform,

$$\begin{aligned} u_\varepsilon(x) &= \frac{1}{(2\pi)^m}\int e^{i\xi\cdot(x-x')}\widehat{\chi}(\varepsilon\xi)u(x')d\xi dx' \\ &= \int \chi_\varepsilon(x-x')u(x')dx' = \chi_\varepsilon * u(x), \end{aligned}$$

with $\chi_\varepsilon(x) = \varepsilon^{-m}\chi(x/\varepsilon)$. Thus, u_ε is a standard regularization of u and the lemma follows. The same proof works for $u(x) \in \mathscr{E}'(\mathbb{R}^m)$ if we understand the appropriate integrals as duality brackets. □

Example 2.1. Choose $c > 0$ so that $\psi(x) = ce^{-|x|^4}$ has integral 1. For $\lambda = 1/4$ we obtain the transform

$$\mathfrak{F}_{\psi,\lambda}u(x, \xi) = c \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')} e^{-|\xi||x-x'|^4} u(x')dx', \quad x, \xi \in \mathbb{R}^m,$$

analogous to the standard FBI but with a different exponential factor (i.e., $e^{-|x|^2}$ replaced by $e^{-|x|^4}$).

Remark 2.1. Although the inversion formula holds for any $\lambda > 0$, we will only obtain useful FBI transforms for $0 < \lambda < 1$ as will become clear in the next section.

3. Characterization of the C^∞ Wave Front Set

Let $u \in \mathscr{E}'(\mathbb{R}^m)$. We recall that $(x_0, \xi^0) \notin WF(u)$ if there exist $\phi \in C_c^\infty(\mathbb{R}^m)$ and an open cone $\xi^0 \in \Gamma \subset \mathbb{R}^m$ such that $\phi(x_0) \neq 0$ and

$$\sup_{\Gamma} |\xi|^k \big|(\phi u)\hat{\ }(\xi)\big| < \infty, \quad k = 1, 2, \ldots \tag{3.1}$$

We fix $0 < \lambda < 1$ and $\psi \in \mathscr{S}(\mathbb{R}^m)$ and set

$$\mathfrak{F}u(t, \xi) = \mathfrak{F}_{\psi,\lambda}u(t, \xi) = \langle u(x'), e^{i\xi\cdot(t-x')}\psi\big(|\xi|^\lambda(t - x')\big)\rangle, \quad t, \xi \in \mathbb{R}^m,$$

which is a continuous function of t and ξ, smooth for $\xi \neq 0$. Note that $\mathfrak{F}u(t, \xi)$ may be regarded, for fixed $\xi \in \mathbb{R}^m$, as the convolution of $u(x)$ with the function $x \mapsto e^{\xi\cdot ix}\psi(|\xi|^\lambda x)$ which is smooth and belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^m)$ if $\xi \neq 0$. Furthermore, denoting by N the order of u and recalling that $0 < \lambda < 1$ we have an estimate

$$\begin{aligned} |\mathfrak{F}u(t, \xi)| &\le C \sum_{|\alpha|\le N} \sup_{x'} \big|D_{x'}^\alpha\big(e^{i\xi\cdot(t-x')}\psi\big(|\xi|^\lambda(t - x')\big)\big)\big| \\ &\le C'(1 + |\xi|)^N, \quad t, \xi \in \mathbb{R}^m. \end{aligned}$$

If $u(x) \in C_c^0(\mathbb{R}^m)$ we have $N = 0$ in the above estimate, so $\mathfrak{F}u(t, \xi)$ is bounded. If $u(x) \in C_c^\infty(\mathbb{R}^m)$, the formula

$$\xi^\alpha \mathfrak{F}u(t, \xi) = \langle D_{x'}^\alpha\big[\psi\big(|\xi|^\lambda(t - x')\big)u(x')\big], e^{i\xi\cdot(t-x')}\rangle, \quad t, \xi \in \mathbb{R}^m,$$

may be used to show that $\mathfrak{F}u(t, \xi)$ decreases rapidly as $\xi \to \infty$, uniformly in t, since the right hand side is bounded by $C_\alpha(1 + |\xi|)^{\lambda|\alpha|}$ and $0 < \lambda < 1$.

For t away from the support of u we get better estimates. More precisely, if the support of u is contained in $B_R = B(0, R)$ we have $|t - x'| \ge |t|/2$ for $|t| \ge 2R$, while $|t - x'| \ge \epsilon$ for $x' \in \operatorname{supp} u$ if $\operatorname{dist}(t, \operatorname{supp} u) \ge \epsilon$. It follows that for some constant $c_\epsilon > 0$, $|t - x'| \ge c_\epsilon(1 + |t|)$ if $x' \in \operatorname{supp} u$ and $\operatorname{dist}(t, \operatorname{supp} u) \ge \epsilon$. Hence, using the fact that ψ decreases rapidly at infinity, we obtain for some constants $C_{k,\epsilon} > 0$,

$$\begin{aligned} |\mathfrak{F}u(t, \xi)| \le C_{k,\epsilon}(1 + |\xi|)^N\big(1 + |\xi|^\lambda(1 + |t|)\big)^{-k}, \\ \text{if } \operatorname{dist}(t, \operatorname{supp} u) \ge \epsilon, \quad \xi \in \mathbb{R}^m, \ k = 1, 2, \ldots, \end{aligned} \tag{3.2}$$

showing that $\mathfrak{F}u(t, \xi)$ decreases rapidly in (t, ξ) off $\operatorname{supp} u \times \mathbb{R}^m$. The obvious commutation formula $D_x \mathfrak{F}u = \mathfrak{F}D_x u$ shows that estimates like (3.2) also hold for the derivatives of $\mathfrak{F}u$ of any order with respect x.

Let's now look at the inversion formula, which is obtained as the limit as $\varepsilon \searrow 0$ of the functions

$$u_\varepsilon(x) = \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\varepsilon\xi) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} \, dt \, d\xi \tag{3.3}$$

and is valid for $u \in \mathscr{E}'(\mathbb{R}^m)$. Fix x_0 in the support of u and consider a smooth partition of unity $1 = \chi_1(t - x_0) + \chi_2(t - x_0) + \chi_3(t - x_0)$, $t \in \mathbb{R}^m$, satisfying $0 \leq \chi_j \leq 1$ and

$$\begin{aligned} &\operatorname{supp} \chi_1 \subset \{|t| \leq 2a\}, \\ &\operatorname{supp} \chi_2 \subset \{a \leq |t| \leq A + 1\}, \\ &\operatorname{supp} \chi_3 \subset \{A \leq |t|\}, \end{aligned}$$

for constants $0 < a < A$ to be chosen later. This partition of unity may be used to decompose the integral (3.3) into three integrals, so as to obtain

$$\begin{aligned} u_\varepsilon(x) &= u_{1,\varepsilon}(x) + u_{2,\varepsilon}(x) + u_{3,\varepsilon}(x) \quad \text{where for } j = 1, 2, 3, \\ u_{j,\varepsilon}(x) &= \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\varepsilon\xi) \chi_j(t - x_0) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} \, dt \, d\xi, \end{aligned}$$

If we take A large enough, $\chi_3(t - x_0)$ will be supported away from the support of u where (3.2) holds. Therefore, for k large enough,

$$\begin{aligned} |u_{3,\varepsilon}(x)| &\leq C_k \int_{\mathbb{R}^m} (1 + |\xi|)^N d\xi \int_{|t| \geq 2R} (1 + |\xi|^\lambda |t|)^{-k} |\xi|^{\lambda m} dt \\ &\leq C_k \int_{\mathbb{R}^m} (1 + |\xi|)^N (1 + |\xi|^\lambda)^{m-k+1} d\xi \leq C_k', \end{aligned}$$

showing that $u_{3,\varepsilon}(x)$, $0 < \varepsilon \leq 1$, is uniformly bounded. Similarly, we may prove that $|D_x^\alpha u_{3,\varepsilon}(x)| \leq C_\alpha$, $\alpha \in \mathbb{Z}_+^m$, $0 < \varepsilon \leq 1$.

In order to bound $u_{2,\varepsilon}(x)$ we write

$$u_{2,\varepsilon}(x) = \left\langle u(x'), \int_{\mathbb{R}^m \times \mathbb{R}^m} \chi_2(t - x_0) e^{i\xi\cdot(x-x')} \sigma(\varepsilon\xi) \psi\big(|\xi|^\lambda (t - x')\big) |\xi|^{\lambda m} d\xi \, dt \right\rangle$$

and are led to consider the function

$$x' \mapsto v_\varepsilon(x', x, t) = \chi_2(t - x_0) \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')} \sigma(\varepsilon\xi) \psi\big(|\xi|^\lambda (t - x')\big) |\xi|^{\lambda m} \, d\xi.$$

On the support of $\chi_2(t - x_0)$, the parameter t is restricted to $a \leq |t - x_0| \leq A + 1$, and we will further restrict x by imposing that $|x - x_0| < a/2$. Thus, for those values of x and t, we have

$$\frac{a}{2} \leq |x - t| \leq |x' - t| + |x - x'| \leq 2 \max(|x' - t|, |x - x'|), \quad x' \in \mathbb{R}^n. \tag{3.4}$$

Fix x and t and let's find a bound for $v_\varepsilon(x', x, t)$ that does not depend on ε. It is enough to majorize

$$\int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')}\sigma(\varepsilon\xi)\eta(\xi)\psi\big(|\xi|^\lambda(t-x')\big)|\xi|^{\lambda m}\,d\xi \tag{3.5}$$

with $\eta(\xi)$ smooth, supported in $|\xi| \geq 1$ such that $\eta(\xi) \equiv 1$ for $|\xi| \geq 2$. If $|x' - t| \geq a/4$, we have $|\psi\big(|\xi|^\lambda(t-x')\big)| \leq C_k(1 + a|\xi|^\lambda/4)^{-k}$, $k \in \mathbb{Z}_+$, so choosing k large we get, $|v_\varepsilon(x', x, t)| \leq C\chi_2(t - x_0)$, with C independent of ε. If, on the other hand, $|x' - t| \leq a/4$, (3.4) shows that $|x - x'| \geq a/4$ so writing $e^{i\xi\cdot(x-x')} = |x - x'|^{-2k}(-\Delta_\xi)^k e^{i\xi\cdot(x-x')}$ and integrating by parts (here Δ_ξ denotes the Laplace operator in the variables $\xi_1, \ldots, \xi_m$) we may bound the integrand in (3.5) by $C|\xi|^{\lambda m + 2k(\lambda-1)}$, with C independent of $0 < \varepsilon \leq 1$, which is integrable for large k (here we use once again that $\lambda < 1$). Summing up, we have shown that

$$|v_\varepsilon(x', x, t)| \leq C\chi_2(t - x_0), \quad x' \in \mathbb{R}^m, |x - x_0| < a/2,$$

and a similar reasoning shows that

$$|D^\alpha_{x'} v_\varepsilon(x', x, t)| \leq C_\alpha\chi_2(t - x_0), \quad x' \in \mathbb{R}^m, \ |x - x_0| < a/2.$$

Integrating this with respect to t we see that $u_{2,\varepsilon}(x)$ is defined through the action of $u(x')$ on a family of functions depending on some parameters x and ε. Furthermore, this family is bounded in $C^\infty(\mathbb{R}^m)$ if $|x - x_0| < a/2$ and $0 < \varepsilon \leq 1$, showing that $|u_{2,\varepsilon}(x)| \leq C$, $|x - x_0| < a/2$, $0 < \varepsilon \leq 1$. Similarly, $|D^\alpha_x u_{2,\varepsilon}(x)| \leq C_\alpha$, $|x - x_0| < a/2$, $0 < \varepsilon \leq 1$. We may now pick up a sequence $\varepsilon_k \searrow 0$ such that $u_{2,\varepsilon_k}(x) + u_{3,\varepsilon_k}(x) \to w(x)$ in C^∞ for $|x - x_0| < a/2$. This shows that

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon = \lim_{\varepsilon_k \to 0} u_{1,\varepsilon_k} + w(x) \doteq u_1(x) + w(x), \quad |x - x_0| < a/2,$$

in particular, $(x_0, \xi^0) \in WF(u)$ if and only if $(x_0, \xi^0) \in WF(u_1)$.

Theorem 3.1. *Let $u \in \mathscr{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$ and $|\xi^0| = 1$.*

i) *Assume that there is a ball $B_\delta = B(x_0, \delta) \subset \mathbb{R}^m$ and an open cone $\Gamma \subset \mathbb{R}^m \backslash \{0\}$ containing ξ^0 such that*

$$\sup_{B\times\Gamma} |\xi|^k \, |\mathfrak{F}u(t, \xi)| < \infty, \quad k = 1, 2, \ldots, \tag{3.6}$$

holds. Then $(x_0, \xi^0) \notin WF(u)$.

ii) *Conversely, if $(x_0, \xi^0) \notin WF(u)$ then* (3.6) *holds for some $B_\delta = B(x_0, \delta)$ and some open cone $\Gamma \ni \xi^0$.*

Proof. In the proof of i), we will assume without loss of generality that $x_0 = 0$. According to the considerations above, to prove i) we need only show that $(0, \xi^0) \notin WF(u_1)$. Note that by the choice of $\chi_1(t)$

$$u_{1,\varepsilon}(x) = \int_{\{|t|\leq 2a\}\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\varepsilon\xi)\chi_1(t)\mathfrak{F}u(t, \xi)|\xi|^{\lambda m}\,dt\,d\xi,$$

and we are free to choose $a > 0$ as small as we wish, so we may assume that $2a < \delta$, i.e., $\operatorname{supp} \chi_1 \subset B_\delta$. Choose functions $\alpha(\xi)$, $\beta(\xi) \in C^\infty(\mathbb{R}^m)$ such that

(i) $\alpha(\xi)$ and $\beta(\xi)$ are positively homogeneous of degree 0 for $|\xi| \geq 1$;
(ii) $\alpha(\xi) + \beta(\xi) = 1$, $\xi \in \mathbb{R}^m$;
(iii) $\operatorname{supp} \alpha \cap \{|\xi| \geq 1\} \subset \Gamma$;
(iv) there exists an open subcone $\xi^0 \in \Gamma_1 \subset \Gamma$ such that $\operatorname{supp} \beta \cap \overline{\Gamma}_1 = \emptyset$.

Next, we write $u_{1,\varepsilon}(x) = A_\varepsilon(x) + B_\varepsilon(x)$ with

$$\text{(a)} \quad A_\varepsilon(x) = \int_{B_\delta \times \Gamma} e^{i\xi\cdot(x-t)}\, \alpha(\xi)\sigma(\varepsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}\, dt\, d\xi,$$

$$\text{(b)} \quad B_\varepsilon(x) = \int_{B_\delta \times \mathbb{R}^m} e^{i\xi\cdot(x-t)}\, \beta(\xi)\sigma(\varepsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}\, dt\, d\xi.$$

Using estimates (3.6) and letting $\varepsilon \to 0$ in (a) we see that $A_\varepsilon(x)$ converges in $C^\infty(\mathbb{R}^m)$ to a smooth function $A(x)$. Hence, B_ε converges in $\mathscr{D}'(\{|x| < a/2\})$ to a distribution $B(x)$ and we must show that $(0, \xi^0) \notin WF(B)$. We may write

$$\begin{aligned} B_\varepsilon(x) &= \left\langle u(x'), \int_{B\times\mathbb{R}^m} \beta(\xi)\chi_1(t)e^{i\xi\cdot(x-x')}\sigma(\varepsilon\xi)\psi\big(|\xi|^\lambda(t-x')\big)|\xi|^{\lambda m}\, d\xi\, dt \right\rangle \\ &= \int_{\mathbb{R}^m} e^{i\xi\cdot x}\beta(\xi)\sigma(\varepsilon\xi)|\xi|^{\lambda m}\left\langle u(x'), e^{-i\xi\cdot x'} \int_B \chi_1(t)\psi\big(|\xi|^\lambda(t-x')\big)dt \right\rangle d\xi \end{aligned}$$

which may be written as

$$B_\varepsilon(x) = \int_{\mathbb{R}^m} e^{i\xi\cdot x}\beta(\xi)\sigma(\varepsilon\xi)|\xi|^{\lambda m}\Big\langle u(x'), e^{-i\xi\cdot x'}b(x',\xi)\Big\rangle d\xi. \tag{3.7}$$

Notice that $b(x',\xi) = \int \chi_1(t)\psi\big(|\xi|^\lambda(t-x')\big)dt$ and its derivatives with respect to x' have tempered growth in ξ uniformly in x', so for fixed $0 < \varepsilon \leq 1$

$$\beta(\xi)\sigma(\varepsilon\xi)|\xi|^{\lambda m}\langle u(x'), e^{-i\xi\cdot x'}b(x',\xi)\rangle$$

is an integrable function and (3.7) means that

$$\widehat{B}_\varepsilon(\xi) = (2\pi)^m\beta(\xi)\sigma(\varepsilon\xi)|\xi|^{\lambda m}\langle u(x'), e^{-i\xi\cdot x'}b(x',\xi)\rangle.$$

This shows that $\widehat{B}_\varepsilon(\xi)$ vanishes on Γ_1. Let $\phi(x) \in C_c^\infty\big(\{|x| < a/2\}\big)$ satisfy $\phi(0) = 1$ and let $\eta(\xi) \in C^\infty(\mathbb{R}^m)$ be positively homogeneous for $|\xi| \geq 2$, with $\eta(2\xi^0) = 1$ and $\operatorname{supp}\eta \cap \operatorname{supp}\beta = \emptyset$. Then the pseudodifferential operator $p(x,D) \in S^0_{1,0}$ with symbol $p(x,\xi) = \phi(x)\eta(\xi)$ is microlocally elliptic at $(0,\xi^0)$ and $p(x,D)B_\varepsilon = 0$. We now consider a properly supported pseudodifferential operator $P(x,D)$ in $\{|t| < a/2\}$ with symbol $p(x,\xi)$ (so $P(x,D) = p(x,D) + R(x,D)$, with $R(x,D)$ a regularizing operator) and we let $\varepsilon \searrow 0$ to get $P(x,D)B_\varepsilon \to P(x,D)B = R(x,D)B \in C^\infty$. This implies that $(0,\xi^0) \notin WF(B)$.

We now prove ii). We may write $u = \phi u + (1-\phi)u = u_1 + u_2$ with $\phi \in C_c^\infty(\mathbb{R}^m)$ such that (3.1) holds for some open cone Γ around ξ^0. We are allowed to choose $\phi(x) \equiv 1$ for $|x - x_0|$ small and we do so, in particular, $x_0 \notin \operatorname{supp} u_2$. Since $\mathfrak{F}u_2(t,\xi)$ decreases rapidly off the support of u_2 it is enough to focus on the decay of $\mathfrak{F}u_1(t, \xi)$. In other words, we may assume from the start and without loss of generality that $\hat{u}(\xi)$ decreases rapidly on Γ. Choose functions $\alpha(\xi)$, $\beta(\xi) \in C^\infty(\mathbb{R}^m)$ such that

(i) $\alpha(\xi)$ and $\beta(\xi)$ are positively homogeneous of degree 0 for $|\xi| \geq 1$;
(ii) $\alpha(\xi) + \beta(\xi) = 1$, $\xi \in \mathbb{R}^m$;
(iii) $\operatorname{supp} \alpha \cap \{|\xi| \geq 1\} \subset \Gamma$;
(iv) there exists an open subcone $\xi^0 \in \Gamma_1 \subset \Gamma$ such that $\operatorname{supp} \beta \cap \overline{\Gamma}_1 = \emptyset$.

By the Fourier inversion formula,

$$\begin{aligned} u(x') &= \frac{1}{(2\pi)^m} \int e^{ix'\cdot\eta} \hat{u}(\eta) d\eta \\ &= \frac{1}{(2\pi)^m} \int e^{ix'\cdot\eta} \alpha(\eta) \hat{u}(\eta) d\eta + \frac{1}{(2\pi)^m} \int e^{ix'\cdot\eta} \beta(\eta) \hat{u}(\eta) d\eta \\ &= u_1(x') + u_2(x'). \end{aligned}$$

Since $\hat{u}(\xi)$ decreases rapidly on Γ, (iii) implies that the integral that defines u_1 is absolutely convergent and repeated differentiation under the integral sign shows that $u_1 \in C^\infty(\mathbb{R}^m)$. Note also that $u_1 \in L^\infty(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$. Since u is of compact support, $|\mathfrak{F}u(x, \xi)|$ is bounded for $|\xi| \leq 1$ and so in the rest of the proof we may assume that $|\xi| \geq 1$. Let u_{11}, u_{12} be smooth functions such that $u_1 = u_{11} + u_{12}$, $u_{11}(x)$ has a compact support, and $u_{12}(x)$ is supported on $\{x : |x| \geq 2a\}$ for some $a > 0$. Clearly $\mathfrak{F}u_{11}(t, \xi)$ decreases rapidly. Consider next

$$\mathfrak{F}u_{12}(x, \xi) = \int e^{i(x-x')\cdot\xi} \psi(|\xi|^\lambda (x - x')) u_{12}(x') dx'.$$

For $|x| \leq a$ and x' in the support of u_{12}, and for any integer $k \geq 1$, there is a constant $C_k > 0$ such that

$$|\psi(|\xi|^\lambda (x - x'))| \leq C_k |\xi|^{-k\lambda} |x'|^{-k} \quad \text{whenever } |\xi| \geq 1.$$

Hence for $|x| \leq a$ and $|\xi| \geq 1$,

$$|\mathfrak{F}u_{12}(x, \xi)| \leq C_k |\xi|^{-k\lambda} \int \frac{u_{12}(x')}{|x'|^k} dx'.$$

Hence $\mathfrak{F}u_1(x, \xi)$ decays rapidly for x near the origin and it is enough to focus on $\mathfrak{F}u_2(t, \xi)$. Take a function $\phi(x) \in C_c^\infty(\mathbb{R}^m)$ with $\int \phi(x) dx = 1$. For $\epsilon > 0$ let

$$\mathfrak{F}u_{2,\epsilon}(x, \xi) = \int \int e^{ix\cdot\xi} e^{ix'\cdot(\eta-\xi)} \psi(|\xi|^\lambda (x - x')) \hat{\phi}(\epsilon\eta) \beta(\eta) \hat{u}(\eta) d\eta \, dx'. \tag{3.8}$$

By (i) and (iv), there is a constant $c > 0$ such that

$$\xi \in \Gamma_1, \quad \eta \in \operatorname{supp} \beta \Longrightarrow |\xi - \eta| \geq c|\xi|. \tag{3.9}$$

Using the formula $(-\Delta_{x'})^k\big(e^{ix'\cdot(\eta-\xi)}\big) = |\xi-\eta|^{2k}e^{ix'\cdot(\eta-\xi)}$ to integrate by parts in (3.8), we get for large k in view of (3.9):

$$|\mathfrak{F}u_{2,\epsilon}(x,\xi)| \leq C_k \int_{|\xi-\eta|\geq c|\xi|} \frac{|\xi|^{2k\lambda}}{|\xi-\eta|^{2k}}\, d\eta \leq C_k'|\xi|^{m+2k(\lambda-1)}, \quad \xi \in \Gamma_1$$

where the constants C_k and C_k' are independent of ϵ. Since $\mathfrak{F}u_{2,\epsilon}(x,\xi)$ converges to $\mathfrak{F}u_2(x,\xi)$, we conclude that $\mathfrak{F}u_2(x,\xi)$ decreases rapidly as $|\xi|\to\infty$ in Γ_1. $\square$

4. Characterization of the Analytic Wave Front Set

Taking as a starting point the transform in Example 1.1 we consider next a subclass of FBI transforms that may be used to characterize not only microlocal smoothness but also microlocal analyticity as the usual FBI does. Let $p(x)$, $x\in\mathbb{R}^m$, be a real, homogeneous, positive elliptic polynomial of degree $2k$, $k\in\mathbb{N}$, i.e.,

$$p(x) = \sum_{|\alpha|=2k} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R},$$

satisfies

$$c|x|^{2k} \leq p(x) \leq C|x|^{2k}$$

for some positive constants $0 < c \leq C$. Note that $p(\lambda x) = \lambda^{2k}p(x)$, $x\in\mathbb{R}^m$, $\lambda\in\mathbb{R}$.

We now take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = 1/(2k)$ as a parameter and consider the FBI transform

$$\mathfrak{F}u(t,\xi) = c_p\int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')}e^{-|\xi|p(t-x')}u(x')dx', \quad x,\xi\in\mathbb{R}^m.$$

We have dropped any reference to ψ and λ in the notation of $\mathfrak{F}$ because they will be kept fixed throughout this section. The inversion formula is thus

$$u(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\,\sigma(\varepsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{m/2k}dt\,d\xi. \tag{4.1}$$

We need to recall a fact [6, Theorem V.2.9] and two definitions. In what follows, for an open set $\Omega \in \mathbb{C}^m$, $\mathcal{O}(\Omega)$ will denote the space of holomorphic functions on Ω.

Theorem 4.1. *Any $u\in\mathcal{E}'(\mathbb{R}^m)$ can be expressed as a finite sum $\sum_{j=1}^n bf_j$ where each $f_j\in\mathcal{O}(\mathbb{R}^m+i\Gamma_j')$ for some cones $\Gamma_j'\subseteq\mathbb{R}^m$, and the f_j are of tempered growth.*

Definition 4.2. Let $u\in\mathcal{D}'(\mathbb{R}^m)$, $x_0\in\mathbb{R}^m$, $\xi^0\in\mathbb{R}^m\backslash\{0\}$. We say that u is *microlocally analytic* at (x_0,ξ^0) if there exist a neighborhood V of x_0, cones $\Gamma^1,\ldots,\Gamma^N$ in $\mathbb{R}^m\backslash\{0\}$ and holomorphic functions $f_j\in\mathcal{O}(V+i\Gamma_\delta^j)$ (for some $\delta>0$) of tempered growth such that $u=\sum_{j=1}^N bf_j$ near x_0 and $\xi^0\cdot\Gamma^j<0\ \forall j$.

Here we are using the notation $\Gamma_\delta = \{v\in\Gamma : |v|<\delta\}$, and bf_j denotes the boundary value of f_j.

Definition 4.3. The *analytic wave front set* of a distribution u, denoted $\mathrm{WF}_a(u)$ is defined by

$$\mathrm{WF}_a(u) = \{(x, \xi) : u \text{ is not microlocally analytic at } (x, \xi)\}.$$

The main result of this section is

Theorem 4.4. *Let* $u \in \mathscr{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$ *and* $|\xi^0| = 1$. *Then* $(x_0, \xi^0) \notin WF_a(u)$ *if and only if there are a neighborhood* V *of* x_0, *a conic neighborhood* Γ *of* ξ^0 *and constants* $c_1, c_2 > 0$, *such that*

$$|\mathfrak{F}u(t, \xi)| \leq c_1 e^{-c_2|\xi|}, \quad (t, \xi) \in V \times \Gamma. \tag{4.2}$$

Proof. We may assume that $x_0 = 0$. Suppose $(0, \xi^0) \notin WF_a(u)$. Then to establish (4.2) we may assume, without loss of generality, that u is the boundary value of a function f which is holomorphic on a truncated wedge $U + i\Gamma_\delta$ where $\Gamma_\delta = \{v \in \Gamma : |v| < \delta\}$, Γ is an open cone in $\mathbb{R}^m$ satisfying

$$v \cdot \xi^0 < 0, \quad \forall v \in \Gamma \tag{4.3}$$

and U is a neighborhood of the origin in $\mathbb{R}^m$. Let $r > 0$ so that $B_{2r} = \{x \in \mathbb{R}^m : |x| < 2r\} \subset\subset U$, and $\psi \in C_0^\infty(B_{2r})$, $\psi \equiv 1$ on B_r. Fix $v \in \Gamma_\delta$. Let

$$Q(x', \xi, x) = i\xi \cdot (x' - x) - |\xi|\, p(x' - x).$$

Modulo an exponential decay for x' near the origin and $\xi \in \mathbb{R}^m$, we have:

$$\begin{aligned}
\mathfrak{F}u(x', \xi) &\simeq \mathfrak{F}(\psi u)(x', \xi) \\
\mathfrak{F}(\psi u)(x', \xi) &= \lim_{t \to 0^+} \int_{B_{2r}} e^{Q(x', \xi, x)} \psi(x) f(x + itv)\, dx \\
&= \lim_{t \to 0^+} \int_{B_{2r}} e^{Q(x', \xi, x + itv)} \psi(x) f(x + itv)\, dx.
\end{aligned}$$

For $\lambda > 0$ to be determined later, and $0 < t < \lambda$, let

$$D_t = \{x + isv \in \mathbb{C}^m : x \in B_{2r},\ t \leq s \leq \lambda\}.$$

Consider the m-form

$$e^{Q(x', \xi, z)} \psi(z) f(z)\, dz_1 \wedge \cdots \wedge dz_m,$$

where for $z = x + iy$, $\psi(z) = \psi(x)$. Since $\psi \in C_0^\infty(B_{2r})$, by Stokes' theorem,

$$\begin{aligned}
&\int_{B_{2r}} e^{Q(x', \xi, x + itv)} \psi(x) f(x + itv)\, dx \\
&\quad = \int_{B_{2r}} e^{Q(x', \xi, x + i\lambda v)} \psi(x) f(x + i\lambda v)\, dx \\
&\quad + \sum_{j=1}^{m} \int_{D_t} e^{Q(x', \xi, x + isv)} \frac{\partial \psi}{\partial \bar{z}_j}(x) f(x + isv)\, d\bar{z}_j \wedge dz_1 \wedge \cdots \wedge dz_m.
\end{aligned} \tag{4.4}$$

We will estimate the two integrals on the right hand side of equation (4.4). Since $\xi^0 \cdot v < 0$, we can get an open cone $\mathscr{C}$ containing ξ^0 such that for some $c_0 > 0$, $\xi \cdot v \le -c_0|\xi||v|$ whenever $\xi \in \mathscr{C}$. To estimate the first integral, observe that for $\xi \in \mathscr{C}$,

$$\begin{aligned}\Re Q(x', \xi, x + i\lambda v) &= \lambda(\xi \cdot v) - c|\xi|(|x' - x|^{2k} + O(\lambda^2)|v|^2) \\ &\le -c_0\lambda|v||\xi| + O(\lambda^2)|\xi|.\end{aligned}$$

We can therefore choose λ small enough such that in the first integral,

$$\Re Q(x', \xi, x + i\lambda v) \le -\frac{c_0}{2}\lambda|v||\xi| \quad \forall \xi \in \mathscr{C}. \tag{4.5}$$

For the second integral, we have

$$\begin{aligned}\Re Q(x', \xi, x + isv) &= s(\xi \cdot v) - c|x' - x|^{2k}|\xi| + O(\lambda^2)|\xi| \\ &\le -c|x' - x|^{2k}|\xi| + O(\lambda^2)|\xi|.\end{aligned}$$

In this integral, when $x \in \operatorname{supp}(\partial\psi/\partial\bar{z}_j)$, $|x| \ge r$ and so for $|x'| \le r/2$ and λ small enough, we can get $\rho > 0$ such that

$$\Re Q(x', \xi, x + isv) \le -\rho|\xi| \quad \text{whenever } \xi \in \mathscr{C}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it follows that there exist constants $a, b > 0$ independent of t, a neighborhood V of the origin, and a conic neighborhood $\mathscr{C}$ of ξ^0 such that

$$\left|\int_{B_{2r}} e^{Q(x',\xi,x+itv)}\psi(x)f(x + itv)dx\right| \le ae^{-b|\xi|} \quad \text{for all } (x', \xi) \in V \times \mathscr{C}$$

which implies the same decay for $\mathfrak{F}u(x', \xi)$ on $V \times \mathscr{C}$ and proves (4.2).

To prove the converse implication, suppose now that

$$|\mathfrak{F}u(t, \xi)| \le c_1 e^{-c_2|\xi|} \quad \text{for } (x, \xi) \in V \times \Gamma$$

for some conic neighborhood $V \times \Gamma$ of $(0, \xi^0)$. We start by invoking the inversion formula (4.1) and write $u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x)$, $x \in \mathbb{R}^m$, with

$$u_\varepsilon(z) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(z-t)}\sigma(\varepsilon\xi)\mathfrak{F}u(t, \xi)|\xi|^{m/2k}\, dt\, d\xi, \quad z \in \mathbb{C}^m, \tag{4.7}$$

so $u_\varepsilon(z)$ is an entire holomorphic function of z. Write the integral in (4.7) as the sum of two integrals

$$u_\varepsilon(z) = u_0^\varepsilon(z) + u_1^\varepsilon(z)$$

by decomposing the domain of integration $\mathbb{R}^m \times \mathbb{R}^m$ into two sets:

$$\begin{aligned}u_0^\varepsilon(z) &= \text{the integral over}\{(t, \xi) : |t| \le a,\ \xi \in \mathbb{R}^m\}, \\ u_1^\varepsilon(z) &= \text{the integral over}\{(t, \xi) : |t| \ge a,\ \xi \in \mathbb{R}^m\}.\end{aligned}$$

We will prove below in Lemma 4.1 that for any choice of $a > 0$ sufficiently small, the entire functions $u_1^\varepsilon(z)$ converge to a holomorphic function on some ball $B_\delta \subset \mathbb{C}^m$ as $\varepsilon \to 0$. Thus, assuming this fact momentarily and choosing $\sigma(\xi) = e^{-|\xi|^2}$, modulo the boundary value of a holomorphic function, we have

$$u(x) \simeq \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^m} \int_{\{t:|t|\le a\}} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2}\, \mathfrak{F}u(t,\xi)|\xi|^{m/2k}\, dt\, d\xi$$

where we will assume that $\{t \in \mathbb{R}^m : |t| \le a\} \subseteq V$. Let $\mathscr{C}_j$, $1 \le j \le N$ be open, acute cones such that

$$\mathbb{R}^m = \bigcup_{j=1}^N \overline{\mathscr{C}_j} \quad \text{and} \quad \mathscr{C}_j \cap \mathscr{C}_k$$

has measure zero for $j \ne k$. We can arrange it so that $\xi^0 \in \mathscr{C}_1$, $\mathscr{C}_1 \subseteq \Gamma$, $\xi^0 \notin \overline{\mathscr{C}_j}$ when $j \ge 2$. For $2 \le j \le N$, let Γ_j be open cones such that

$$\xi^0 \cdot \Gamma_j < 0 \text{ and for some } c > 0,\ v \cdot \xi \ge c|v||\xi|\ \forall v \in \Gamma_j,\ \forall \xi \in \mathscr{C}_j.$$

For each $1 \le j \le N$, and $\varepsilon > 0$, define

$$f_j^\varepsilon(x+iy) = \int_{\mathscr{C}_j} \int_{\{t:|t|\le a\}} e^{i\xi\cdot(x+iy-t)-\varepsilon|\xi|^2} \mathfrak{F}u(t,\xi)|\xi|^{m/2k}\, dt\, d\xi.$$

Observe that for $2 \le j \le N$, each f_j^ε is entire and as $\varepsilon \to 0$, the f_j^ε converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma_j$ to

$$f_j(x+iy) = \int_{\mathscr{C}_j} \int_{\{t:|t|\le a\}} e^{i\xi\cdot(x+iy-t)} \mathfrak{F}u(t,\xi)|\xi|^{m/2k}\, dt\, d\xi$$

which is also holomorphic on $\mathbb{R}^m + i\Gamma_j$. Because of the exponential decay of $\mathfrak{F}u(t,\xi)$ on $\mathscr{C}_1 \times \{t : |t| \le a\}$, in a neighborhood of the origin in $\mathbb{C}^m$, the functions $f_1^\varepsilon(x+iy)$ converge, as $\varepsilon \to 0$, uniformly to

$$f_1(x+iy) = \int_{\mathscr{C}_1} \int_{\{t:|t|\le a\}} e^{i\xi\cdot(x+iy-t)} \mathfrak{F}u(t,\xi)|\xi|^{m/2k}\, dt\, d\xi.$$

It is easy to see that in the sense of distributions, for any j,

$$\lim_{y\to 0} f_j(x+iy) = \lim_{\varepsilon\to 0} f_j^\varepsilon(x).$$

Therefore, modulo the boundary value of a holomorphic function,

$$u(x) = \lim_{\varepsilon\to 0} \sum_{j=1}^N f_j^\varepsilon(x) = \lim_{y\to 0} \sum_{j=1}^N f_j(x+iy).$$

Since for $j \ge 2$ each $f_j(x+iy)$ is of tempered growth on $\mathbb{R}^m + i\Gamma_j$, we conclude that $(0, \xi^0) \notin WF_a(u)$. The proof is finished except for the claim that the functions $u_1^\varepsilon(z)$ converge to a holomorphic function, which will be proved below. □

Lemma 4.1. *There exist $\delta > 0$ and a holomorphic function $U(z) \in \mathcal{O}(B_\delta) \subset \mathbb{C}^m$ such that* $\lim_{\varepsilon\to 0} u_1^\varepsilon(z) = U(z)$, $z \in B_\delta$.

Proof. It is enough to show that there exist $\delta > 0$ and $M > 0$ such that $|u_1^\varepsilon(z)| \le M$ for any $|z| \le \delta$ and $0 < \varepsilon \le 1$. Indeed, in this case $\{u_1^\varepsilon\}_{0<\varepsilon<1}$ will be a normal family on the ball $B_\delta \subset \mathbb{C}^m$ and we will be able to choose a sequence $\varepsilon_k \searrow 0$ so that the sequence $(u_1^{\varepsilon_k})$ converges to a holomorphic function $U(z) \in \mathcal{O}(B_\delta)$.

Write $u_1^\varepsilon(z)$, $z = x + iy$, as the sum of the three functions

$$u_1^\varepsilon(z) = I_2^\varepsilon(z) + I_3^\varepsilon(z) + I_4^\varepsilon(z)$$

by decomposing the domain of integration $\{|t| \ge a\} \times \mathbb{R}$ into three sets:

$$\begin{aligned} I_2^\varepsilon(z) &= \text{the integral over } X_2 \doteq \{(t,\xi) : a \le |t| \le A,\ |\xi| \le 1\},\\ I_3^\varepsilon(z) &= \text{the integral over } X_3 \doteq \{(t,\xi) : |t| \ge A,\ \xi \in \mathbb{R}^m\},\\ I_4^\varepsilon(z) &= \text{the integral over } X_4 \doteq \{(t,\xi) : a \le |t| \le A,\ |\xi| \ge 1\}, \end{aligned}$$

where the constant A will be chosen later. We will show that $|I_j^\varepsilon(z)|$, $j = 1, 2, 3$, remain bounded for $0 < \varepsilon < 1$ and $|z| < \delta$ if δ is small.

Since X_2 is a bounded set and $\mathfrak{F}u$ is a continuous function, it is clear that for, say, $|y| \le 1$, $I_2^\varepsilon(z)$ is bounded by a constant independent of ε.

To bound $I_3^\varepsilon(z)$, pick some $r \ge 1$ such that the ball B_r contains the support of u. Choose $A = 2r$, so for $|t| \ge A$ and $|x'| \le r$, $|t - x'| \ge |t|/2$ and $e^{-|\xi|p(t-x')} \le e^{-c|\xi||t|^{2k}} \le e^{-c|\xi||t|}$. Hence, $|\mathfrak{F}u(t,\xi)| \le Ce^{-|t||\xi|}$ for $|t| \ge A$ and

$$\begin{aligned} |I_3^\varepsilon(z)| &\le C\int_{|t|\ge A} e^{|y||\xi|}e^{-|t||\xi|/2}|\xi|^{m/2k}\,dt\,d\xi\\ &\le C\int_{|t|\ge A} e^{|y||\xi|}e^{-A|\xi|/4}e^{-|t||\xi|/4}|\xi|^{m/2k}\,dt\,d\xi\\ &\le C\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-A|\xi|/4}|\xi|^{-\frac{m(2k-1)}{2k}}\,d\xi \le C' \end{aligned}$$

if $|y| \le A/8$. Finally, to estimate $I_4^\varepsilon(z)$, we write for $z = x + iy \in \mathbb{C}^m$,

$$I_4^\varepsilon(z) = \int_{a\le|t|\le A} dt \int_{|x'|\le r} dx' \int_{|\xi|\ge 1} e^{i(z-x')\cdot\xi - p(t-x')|\xi| - \varepsilon|\xi|^2} u(x')|\xi|^{m/2k}\,d\xi.$$

Let $s > 0$ be a small number to be chosen later. Note that the function $\xi \mapsto \log|\xi|$ has a holomorphic extension $\log\langle\zeta\rangle$ in the region $|\mathrm{Im}\,\zeta| < |\Re\zeta|$, where

$$\log\langle\zeta\rangle = \frac{1}{2}\log\left(\sum_{j=1}^m \zeta_j^2\right) = \log\left(\sum_{j=1}^m \zeta_j^2\right)^{1/2}$$

and an appropriate branch of the log is taken. In particular, the functions $\zeta \mapsto \langle\zeta\rangle^{1/(2k)}$ and $\zeta \mapsto \langle\zeta\rangle$ are holomorphic on that region and the latter extends $\xi \mapsto |\xi|$. For x, x' fixed, we will change the integration in ξ from the m-cycle $\{|\xi| \ge 1\} \subset \mathbb{R}^m \subset \mathbb{C}^m$ to the cycle in $\mathbb{C}^m$ formed by the union of the finite cylinder

$$\{\xi + i\sigma s(x - x') : |\xi| = 1,\ 0 \le \sigma \le 1\}$$

with the graph of the map

$$\zeta(\xi) = \xi + is|\xi|(x - x'), \quad \xi \in \mathbb{R}^m, \ 1 \leq |\xi| < \infty.$$

The result of the integral does not change because the form

$$\langle\zeta\rangle^{m/(2k)} \exp\big(i(z - x') \cdot \zeta - p(t - x')\langle\zeta\rangle - \varepsilon[\zeta]^2\big) d\zeta$$

is exact (we have used the notation $[\zeta]^2 = \sum_{j=1}^m \zeta_j^2$). The integral over the cylinder is easily bounded by a constant independent of ε. To estimate the integral over the second portion of the cycle we observe that the real part of the exponent is, for $|x|, |y| < \delta$,

$$\begin{aligned}
&\Re\big(i(z - x') \cdot \zeta - p(t - x')\langle\zeta\rangle - \varepsilon[\zeta]^2\big) \\
&\quad \leq -\big(|x - x'|^2 s - |y| + p(t - x')\big)|\xi| + \varepsilon|\xi|^2\big(1 - s^2|x - x'|^2\big) \\
&\quad \leq \big(\delta - s|x - x'|^2 - p(t - x')\big)|\xi|
\end{aligned}$$

assuming that we take $s = s(\delta, r)$ sufficiently small to guarantee that $s^2|x - x'|^2 \leq 1/2$ for $|x| \leq \delta$ and $|x'| \leq r$. Note that the function

$$x' \mapsto \delta - s|x - x'|^2 - p(t - x') \leq \delta + \min\big(-s|x - x'|^2, -p(t - x')\big)$$

is clearly bounded by

$$\delta - c|t - x'|^{2k} \leq \delta - c\left(\frac{a}{2}\right)^{2k} \quad \text{for } |x'| \leq a/2$$

(recall that $|t| \geq a$) and by

$$\delta - s(a/4)^2 \quad \text{for } |x'| \geq a/2,$$

provided that we take $\delta < a/4$. Hence, choosing $\delta > 0$ sufficiently small, we may bound the integrand on the second portion of the cycle by $e^{-c|\xi|}$ for some $c > 0$. This shows that $|I_4^\varepsilon(z)| \leq C$ for $|z| < \delta$. Summing up, we have shown that $\sup_{0<\varepsilon\leq 1} |u_1^\varepsilon(x + iy)| < \infty$ if $|x| + |y| < \delta$ provided $\delta > 0$ is small enough, as we wished to prove. □

5. Applications and Examples

Let $x = (x_1, \ldots, x_m)$ and $t = (t_1, \ldots, t_n)$ denote the variables in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let $U \subset \mathbb{R}^n$ be a connected open set and $\varphi = (\varphi_1, \ldots, \varphi_m) : U \to \mathbb{R}^m$ be a Lipschitz continuous map. Let

$$Z(x, t) = (Z_1(x, t), \ldots, Z_m(x, t)), \quad \text{where } Z_j(x, t) = x_j + i\varphi_j(t).$$

Consider the system of associated vector fields on $\mathbb{R}^m \times U$ defined by

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^m \frac{\partial \varphi_k(t)}{\partial t_j} \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq n.$$

Note that $L_j Z_k = 0$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$. Let $B_r(0) \subset \mathbb{R}^m$ denote the open ball of radius $r > 0$ and set $\Omega = B_r(0) \times U$. If $h(x, t)$ is a Lipschitz continuous solution in Ω of the system of equations

$$L_j h = 0, \quad 1 \leq j \leq n, \tag{5.1}$$

we are interested in the analytic wave-front set of the function $h_0(x) = h(x, 0)$ that will be denoted by $WF_a(h_0)$. We will assume that $0 \in U$ and $\varphi(0) = 0$.

Theorem 5.1. *Let* $\xi^0 \in \mathbb{R}^m \backslash 0$, $|\xi^0| = 1$, $t^* \in U \backslash 0$ *and* $\gamma \subset U$ *be a Lipschitz curve with* 0 *and* t^* *as its endpoints satisfying for some* $\epsilon > 0$

(1) $-\varphi(t^*) \cdot \xi^0 \geq \epsilon^7$,
(2) $|\varphi(t)| \leq \epsilon^2$ *for* $t \in \gamma$.

There exists $\epsilon_0 > 0$ *such that if* $0 < \epsilon \leq \epsilon_0$ *and* h *is any Lipschitz continuous solution of* (5.1) *in* $\Omega = B_r(0) \times U$, *then* $(0, \xi^0) \notin WF_a(h_0)$.

Proof. Since we are allowed to shrink $B_r(0)$ and $\epsilon > 0$ will be taken small, there is no loss of generality in assuming that $r^2 = \epsilon$ and we will do so. Let $g \in C_0^\infty(B_r(0))$, $g(x) \equiv 1$ for $|x| \leq \frac{r}{2}$. For $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$, consider the integral

$$F(x, t, \xi) = \int_{\mathbb{R}^m} e^{Q(x,y,t,\xi)} g(y) h(y, t) dy,$$

where

$$Q(x, y, t, \xi) = i(x - Z(y, t)) \cdot \xi - K|\xi|(x - Z(y, t))^4,$$

where $K > 0$ will be determined later. Here for $z \in \mathbb{C}^m$, we have used the notation

$$z^4 = \left(\sum_{j=1}^m z_j^2 \right)^2 .$$

Let

$$I(x, \xi) = \int_{\mathbb{R}^m} \int_\gamma e^{Q(x,y,t,\xi)} L\big(g(y)h(y, t)\big) dt\, dy,$$

where

$$Lf(y, t)dt = \sum_{j=1}^n L_j f(y, t) dt_j$$

is a one-form on U depending on the variable y. Integration by parts and the fact that $L_j Z_k = 0$ lead to

$$I(x, \xi) = F(x, t^*, \xi) - F(x, 0, \xi). \tag{5.2}$$

Note that choosing $p(x) = |x|^4$ and $\psi(x) = e^{-|\xi|p(x)}$ we have $\psi(|\xi|^{1/4}x) = e^{-|\xi|p(x)}$ and we may write

$$F(x, 0, \xi) = \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi}\psi(|\xi|^{1/4}(x-y))g(y)h(y, 0)dy,$$
$$= \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi}e^{-|\xi|p(x-y)}g(y)h(y, 0)dy,$$

which means that $F(x, 0, \xi) = \mathfrak{F}(gh_0)(x, \xi)$ is the FBI transform of gh_0 with $\mathfrak{F}$ in the class considered in Section 4. Hence, to prove the theorem, it will be enough to show that for some constants $C_1 > 0$, $c_2 > 0$, the estimate

$$|F(x, 0, \xi)| \leq C_1 e^{-c_2|\xi|}$$

holds for (x, ξ) in a conic neighborhood of $(0, \xi^0)$ and then the result will follow from Theorem 4.4. By (5.2), such an estimate will follow if we show that $I(x, \xi)$ and $F(x, t^*, \xi)$ decay exponentially as $|\xi| \to \infty$ in some conic neighborhood of $(0, \xi^0)$.

Let $E(x, y, t, \xi) = -\Re Q(x, y, t, \xi)$. Observe that

$$|e^{Q(x,y,t,\xi)}| = e^{-E(x,y,t,\xi)}.$$

We will first estimate $E(x, y, t, \xi)$ at $x = 0$ and $\xi = \xi^0$. We have:

$$E(0, y, t, \xi^0) = -\varphi(t) \cdot \xi^0 + K\Re(Z(y, t)^4). \tag{5.3}$$

Note next that if $z = (z_1, \ldots, z_m)$ with $z_j = x_j + iy_j$ for each j, then

$$\Re(z^2) = \Re\left(\sum_{j=1}^m z_j^2\right) = |x|^2 - |y|^2 \quad \text{and} \quad \Im(z^2) = \Im\left(\sum_{j=1}^m z_j^2\right) = 2(x \cdot y).$$

Therefore, setting $w = z^2 = \sum_{j=1}^m z_j^2$, we have

$$\begin{aligned}\Re(z^4) &= \Re(w^2) \\ &= |\Re w|^2 - |\Im w|^2 \\ &= (|x|^2 - |y|^2)^2 - 4(x \cdot y)^2 \\ &\geq |x|^4 + |y|^4 - 6|x|^2|y|^2.\end{aligned} \tag{5.4}$$

From the latter inequality, it follows that

$$\Re(Z(y, t)^4) \geq |y|^4 + |\varphi(t)|^4 - 6|y|^2|\varphi(t)|^2. \tag{5.5}$$

From (5.3) and (5.5) it follows that

$$E(0, y, t, \xi^0) \geq -\varphi(t) \cdot \xi^0 + K(|y|^4 + |\varphi(t)|^4 - 6|y|^2|\varphi(t)|^2). \tag{5.6}$$

In particular, when $t = t^*$, for any $|y| \leq r$, using the assumptions in the theorem, we have:

$$E(0, y, t^*, \xi^0) \geq \epsilon^7 + K(|y|^4 - 6\epsilon^4|y|^2). \tag{5.7}$$

The expression $|y|^4 - 6\epsilon^4|y|^2$ is a quadratic expression in $s = |y|^2$ which attains a minimum value of $-9\epsilon^8$ and hence we conclude that

$$E(0, y, t^*, \xi^0) \geq \epsilon^7 - 9K\epsilon^8 = \epsilon^7(1 - 9K\epsilon). \tag{5.8}$$

Observe that in the integrand of $F(x, t, \xi)$, since $h(y, t)$ is a solution, and $g \equiv 1$ for $|y| \leq \frac{r}{2}$, the term $L(g(y)h(y, t))$ is supported in the ball $|y| \leq \frac{r}{2}$. We will estimate the quantity $E(0, y, t, \xi^0)$ when $|y| \geq \frac{r}{2}$ and $t \in \gamma$. For such y and t, using (5.6) and the assumptions of the theorem, we get for any $\epsilon > 0$ sufficiently small ($\epsilon < 1/(192)^{1/3}$ will do):

$$\begin{aligned} E(0, y, t, \xi^0) &\geq -\varphi(t) \cdot \xi^{\phi} + K(|y|^4 + |\varphi(t)|^4 - 6|y|^2|\varphi(t)|^2) \\ &\geq -\epsilon^2 + K\left(\frac{r^4}{16} - 6r^2\epsilon^4\right) \\ &= -\epsilon^2 + K\left(\frac{\epsilon^2}{16} - 6\epsilon^5\right) \geq \epsilon^2\left(\frac{K}{32} - 1\right) \end{aligned}$$

and hence, for $K \geq 64$,

$$E(0, y, t, \xi^0) \geq \epsilon^2. \tag{5.9}$$

Thus, if we choose $K = 64$ to grant (5.9) and pick $\epsilon_0 < 1/576$, for any $0 < \epsilon < \epsilon_0$, the right hand side of (5.8) will also be positive. Observe next that since the function $E(x, y, t, \xi)$ is homogeneous of degree one in ξ, (5.8) and (5.9) imply that there is a neighborhood V of the origin in $\mathbb{R}^m$, and an open cone $\Gamma \subset \mathbb{R}^m$ containing ξ^0 such that for some $C > 0$,

$$E(x, y, t^*, \xi) \geq C \quad \forall (x, \xi) \in V \times \Gamma, \quad y \in \operatorname{supp} g, \tag{5.10}$$

and

$$E(x, y, t, \xi) \geq C \quad \forall (x, \xi) \in V \times \Gamma, \quad t \in \gamma \text{ and } |y| \geq \frac{r}{2}. \tag{5.11}$$

The theorem follows from estimates (5.10) and (5.11). □

Remark 5.1. The theorem above holds under the weaker assumption that $h(x, t)$ is just continuous and essentially the same proof works; in that case the integral that defines the auxiliary function $I(x, \xi)$ can be given a sense by integrating by parts. In the case $\varphi(t)$ is smooth, the theorem holds for distribution solutions $h(x, t)$.

Next we indicate why it does not seem to be easy to use the usual FBI in order to prove the preceding theorem. Let

$$F_2(x, t, \xi) = \int_{\mathbb{R}^m} e^{Q_2(x,y,t,\xi)} g(y)h(y, t)dy,$$

where

$$Q_2(x, y, t, \xi) = \sqrt{-1}(x - Z(y, t)) \cdot \xi - K|\xi|[x - Z(y, t)]^2,$$

Here for $z \in \mathbb{C}^m$, we are using the notation

$$[z]^2 = \sum_{j=1}^{m} z_j^2.$$

Let $E_2(x, y, t, \xi) = \Re Q_2(x, y, t, \xi)$. Assuming as before that $|\xi^0| = 1$, we have

$$E_2(x, y, t^*, \xi^0) = -\varphi(t^*) \cdot \xi^0 + K(|x - y|^2 - |\varphi(t^*)|^2).$$

Suppose now that $-\varphi(t^*) \cdot \xi^0 = \epsilon^7$ and $|\varphi(t^*)| = \epsilon^2$ so

$$E_2(x, y, t^*, \xi^0) = \epsilon^7 + K(|x - y|^2 - \epsilon^4).$$

In particular,

$$E_2(x, x, t^*, \xi^0) = \epsilon^7 - K\epsilon^4,$$

a quantity which will be negative unless $K < \epsilon^3$. However, if $K < \epsilon^3$, when $|y| \geq \frac{r}{2}$, the term $E_2(x, y, t, \xi^0)$ may become negative. In fact, this will actually occur if say $\sup_\gamma \varphi(t) \cdot \xi^0 \geq C\epsilon^3$ for some $C > 0$.

Microlocal analyticity results for solutions of the system (5.1) were proved in the work [4]. For comparison with the theorem in this section, we recall the main result, [4, Theorem 1.1].

Theorem 5.2 (Theorem 1.1 in [4]). *Let $\xi^0 \in \mathbb{R}^m \backslash 0$ and assume that there are $t^* \in U \backslash 0$ and a Lipschitz curve $\gamma \subset U$ with 0 and t^* as its endpoints satisfying*:

(1) $-\varphi(t^*) \cdot \xi^0 > 0$,
(2) $sup_{t\in\gamma} |\varphi(t)| < r$,
(3) $|\varphi(t^*)|^2 sup_{t\in\gamma} \varphi(t) \cdot \xi^0 < [r^2 - \sup_{t\in\gamma} |\varphi(t)|^2][-\varphi(t^*) \cdot \xi^0]$.

If h is any Lipschitz continuous solution of (1.1) *in* $\Omega = B_r(0) \times U$, *then* $(0, \xi^0) \notin WF_a(h_0)$ $(h_0(x) = h(x, 0)$.

Observe that in a situation in which conditions (1) and (2) of Theorem 5.1 are fulfilled, conditions (1) and (2) of Theorem 5.2 will be satisfied as well. However, in general, condition (3) of Theorem 5.2 may fail to be satisfied. For example if in addition to the conditions of Theorem 5.1, we have that

$$\sup_\gamma \varphi(t) \cdot \xi^0 = \epsilon^2,$$

then the third condition in Theorem 5.2 will not be satisfied.

Example 5.1. Let $\varphi(t) : (-1, 1) \longrightarrow \mathbb{R}$ be a nondecreasing smooth function, with derivative $\varphi'(t) = b(t) \geq 0$. Consider the vector field

$$L = \frac{\partial}{\partial t} - ib(t)\frac{\partial}{\partial x}$$

defined on $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$. Then L is known to be smoothly globally solvable on Ω. Recall that the vector field L is said to be globally hypoelliptic on Ω

if and only any $u \in \mathscr{D}'(\Omega)$ that satisfies $Lu \in C^\infty(\Omega)$ is necessarily smooth. It is also known that

L is globally hypoelliptic if and only if

(♣) *for every* $-1 < t < 1$ *there exist points* $-1 < t_1 < t < t_2$ *such that* $b(t_1) > 0$ *and* $b(t_2) > 0$.

We will show how Theorem 5.1 can be used to prove that (♣) implies the global hypoellipticity of L. If $Lu = f \in C^\infty(\Omega)$, we may find $v \in C^\infty(\Omega)$ such that $Lv = f$, which is the same as saying that $h = u - v$ satisfies $Lh = 0$. Hence, L is globally hypoelliptic if and only if all distribution solutions h of $Lh = 0$ are smooth. Assume without loss of generality that $\varphi(0) = 0$ and $\sup|\varphi| < 1$. Obviously, (♣) implies that we may find a point $0 < t^* < 1$ such that $\varphi(t^*) > 0$ and since $\varphi(t)$ is monotonic (1) and (2) of Theorem 5.1 are fulfilled with $\xi^0 = -1$, $\epsilon^7 = \varphi(t^*) < \epsilon^2$ and $\gamma = [0, t^*]$ (see Remark 5.1 about distribution solutions). It follows that $(0, -1) \notin WF_a(h_0)$. Similarly, we may find a point $-1 < t_* < 0$ with $\varphi(t_*) < 0$ and conclude that $(0, 1) \notin WF_a(h_0)$. It follows that $h_0(x)$ is real analytic at the origin, say $h_0(x) = \sum_{j=1}^\infty a_j x^j$ near the origin. Consider then the function

$$H(x, t) = \sum_{j=1}^\infty a_j (x + i\varphi(t))^j$$

which satisfies $LH = 0$ near $(0, 0)$ and $H(x, 0) = h_0(x) = h(x, 0)$. By uniqueness in the Cauchy problem for locally integrable vector fields, it follows that $h(x, t) = H(x, t)$ near the origin. The same reasoning may be applied to any point (x_0, t_0) by considering the trace at $t = t_0$ of $h(x, t)$ which leads to the conclusion that $h \in C^\infty(\Omega)$.

Remark 5.2. The situation in Example 5.1 is quite simple and the fact that (♣) implies the global hypoellipticity of L can be (and has been) proved in many different ways (construction of parametrices, propagation of singularities, representation of homogeneous solutions by means of a global first integral, etc.). The point of including it here is to illustrate in a very simple setting that FBI transforms can be powerful tools.

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References

[1] Baouendi, M.S., Chang, C.H., Treves, F. (1983). Microlocal hypo-analyticity and extension of CR functions. *J. Diff. Geom.* 18:331–391.

[2] Baouendi, M.S., Rothschild, L.P. (1987). Normal forms for generic manifolds and holomorphic extension of CR fuctions. *J. Diff. Geom.* 25:431–467.

[3] Baouendi, M.S., Rothschild, L.P., Treves, F. (1985). CR structures with group action and extendability of CR functions. *Invent. Math.* 82:359–396.

[4] Baouendi, M.S., Treves, F. (1982). A microlocal version of Bochner's tube theorem. *Indiana University Math. J.* 31:885–895.
[5] Baouendi, M.S., Treves, F. (1984). About the holomorphic extension of CR functions on real hypersurfaces in complex Euclidean space. *Duke Math. J.* 51:77–107.
[6] Berhanu, S., Cordaro, P.D., Hounie, J. (2008). *An Introduction to Involutive Structures*. Cambridge: Cambridge University Press.
[7] Berhanu, S., Hounie, J. (2011). A generalization of Bochner's extension theorem to rough tubes. *J. Geom. Anal.* 21:455–475.
[8] Boggess, A. (1998). The holomorphic extension of C^k CR functions on tube sub-manifolds. *Annales Polonici Mathmatici* LXX:35–42.
[9] Boivin, A., Dwilewicz, R. (1998). Extension and Approximation of CR Functions on Tube Manifolds. *Trans. Amer. Math. Soc*. 350:1945–1956.
[10] Delort, J.-M. (1992). *F.B.I. Transformation, Second Microlocalization and Semilinear Caustics*. Lecture Notes in Mathematics, Vol. 1522. Berlin: Springer-Verlag.
[11] Eastwood, M.G., Robin Graham, C. (2003). Edge of the wedge theory in hypoanalytic manifolds. *Comm. Part. Diff. Eqs.* 28:2003–2028.
[12] Himonas, A.A. (1986). On analytic microlocal hypoellipticity of linear partial differential operators of principal type. *Comm. Part. Diff. Eqs.* 11:1539–1574.
[13] Himonas, A.A. (1989). Semirigid partial differential operators and microlocal analytic hypoellipticity. *Duke Math. J.* 59:265–287.
[14] Hörmander, L. (1990). *The Analysis of Linear Partial Differential Operators, I.* Classics in Mathematics, Vol. 287. New York: Springer-Verlag.
[15] Lerner, N., Morimoto, Y., Xu, C.-J. (2010). Instability of the Cauchy-Kovalevskaya solution for a class of nonlinear systems. *American J. Math.* 132:99–123.
[16] Merker, J. (1994). Global minimality of generic manifolds and holomorphic extendability of CR functions. *Int. Math. Res. Not*. 8:329–342.
[17] Sato, M., Kawai, T., Kashiwara, M. (1973). *Microfunctions and Pseudodifferential Equations, Hyperfunctions and Pseudo-Differential Equations*. Lecture Notes in Mathematics, Vol. 287. New York: Springer-Verlag.
[18] Sjostrand, J. (1982). Singularites analytiques microlocales [[Microlocal analytic singularitie] *Asterisque* 95:1–166.
[19] Treves, F. (1992). *Hypo-Analytic Structures*. Princeton, NJ: Princeton University Press.
[20] Wunsch, J., Zworski, M. (2001). The FBI transform on compact C^∞ manifolds. *Trans. Amer. Math. Soc*. 353:1151–1167.