# Overview Several Complex Variables

Rafael B. Andrist
Giuseppe Della Sala

CIMPA Summer School 2023: Complex Analysis and Geometry
American University of Beirut

# Outline

1. Origins
2. Domains
3. Stein and Oka manifolds
4. PDE methods

# One complex variable

**Definition**

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$

**Some properties of holomorphic $f$**

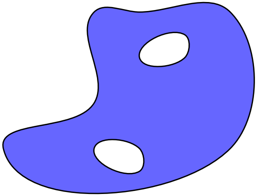
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega : f(z) = 0\}$ has an accumulation point in $\Omega$, then $f = 0$.
- sum, product, composition holomorphic

# One complex variable

## Definition

$f : \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$



### Some properties of holomorphic $f$

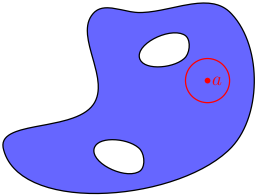
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega : f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# One complex variable

## Definition

$f : \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$




### Some properties of holomorphic $f$

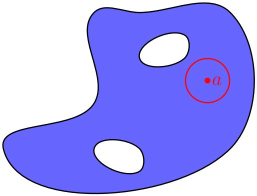
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega : f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# One complex variable

## Definition

$f : \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$




## Some properties of holomorphic $f$

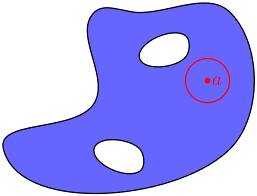
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega \,:\, f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# One complex variable

### Definition

$f : \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$




### Some properties of holomorphic $f$

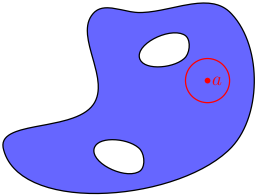
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega \ : \ f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# One complex variable

## Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$




## Some properties of holomorphic $f$

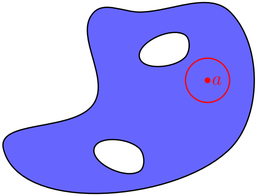
- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega \ :\ f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# One complex variable

## Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z) = \sum_{k=0}^{\infty} c_k \cdot (z-a)^k, \quad a \in \Omega,\ c_k \in \mathbb{C}$$




## Some properties of holomorphic $f$

- maximum principle: If $|f|$ attains a local maximum, then it is constant.
- identity theorem: If $\{z \in \Omega \ : \ f(z) = 0\}$ has an accumulation point in $\Omega$, then $f \equiv 0$.
- sum, product, composition holomorphic

# Several complex variables

## Definition

$f : \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}^n$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n \in \mathbb{N}_0} c_{k_1,\ldots,k_n} \cdot (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} \text{ in } (a_1, \ldots, a_n)$$

## Basic properties

- maximum principle
- identity theorem: if $f$ is locally 0, then $f \equiv 0$.
- sum, product, composition holomorphic

## Definition

A map $f : \Omega \to \mathbb{C}^m$ with $f = (f_1, \ldots, f_m)$ on an open connected subset $\Omega \subset \mathbb{C}^n$ is *holomorphic* if each $f_j : \Omega \to \mathbb{C}$ is a holomorphic function.

# Several complex variables

## Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}^n$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n \in \mathbb{N}_0} c_{k_1,\ldots,k_n} \cdot (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} \text{ in } (a_1,\ldots,a_n)$$

## Basic properties

- maximum principle
- identity theorem: if $f$ is locally $0$, then $f \equiv 0$.
- sum, product, composition holomorphic

## Definition

A map $f\colon \Omega \to \mathbb{C}^m$ with $f = (f_1,\ldots,f_m)$ on an open connected subset $\Omega \subset \mathbb{C}^n$ is *holomorphic* if each $f_j\colon \Omega \to \mathbb{C}$ is a holomorphic function.

# Several complex variables

## Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}^n$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n\in\mathbb{N}_0} c_{k_1,\ldots,k_n}\cdot(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n} \text{ in } (a_1,\ldots,a_n)$$

## Basic properties

- maximum principle
- identity theorem: if $f$ is locally $0$, then $f\equiv 0$.
- sum, product, composition holomorphic

## Definition

A map $f\colon \Omega \to \mathbb{C}^m$ with $f=(f_1,\ldots,f_m)$ on an open connected subset $\Omega \subset \mathbb{C}^n$ is *holomorphic* if each $f_j\colon \Omega \to \mathbb{C}$ is a holomorphic function.

# Several complex variables

### Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}^n$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n \in \mathbb{N}_0} c_{k_1,\ldots,k_n} \cdot (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} \text{ in } (a_1, \ldots, a_n)$$

### Basic properties

- maximum principle
- identity theorem: if $f$ is locally $0$, then $f \equiv 0$.
- sum, product, composition holomorphic

### Definition

A map $f\colon \Omega \to \mathbb{C}^m$ with $f = (f_1, \ldots, f_m)$ on an open connected subset $\Omega \subset \mathbb{C}^n$ is *holomorphic* if each $f_j\colon \Omega \to \mathbb{C}$ is a holomorphic function.

# Several complex variables

## Definition

$f\colon \Omega \to \mathbb{C}$ on a connected open subset $\Omega \subseteq \mathbb{C}^n$ is *holomorphic*, if it can be developed locally in a uniformly convergent complex power series.

$$f(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n \in \mathbb{N}_0} c_{k_1,\ldots,k_n} \cdot (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} \text{ in } (a_1, \ldots, a_n)$$

## Basic properties

- maximum principle
- identity theorem: if $f$ is locally $0$, then $f \equiv 0$.
- sum, product, composition holomorphic

## Definition

A map $f\colon \Omega \to \mathbb{C}^m$ with $f = (f_1, \ldots, f_m)$ on an open connected subset $\Omega \subset \mathbb{C}^n$ is *holomorphic* if each $f_j\colon \Omega \to \mathbb{C}$ is a holomorphic function.

# Hartogs' Phenomenon

A *Reinhardt domain* $\Omega \subseteq \mathbb{C}^n$ is such that
$(z_1, \dots, z_n) \in \Omega \implies (\lambda_1 z_1, \dots, \lambda_n z_n) \in \Omega$ for all $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.
For $n = 1$, $\Omega$ is an annulus or a disk. In general, $\Omega$ is a union of tori.
Every holomorphic function on $\Omega$ has a unique Laurent series expansion.

**Theorem (Hartogs 1906)**

*Let $K \subset \mathbb{D}^n$ be a compact subset of the open unit polydisk. If $\mathbb{D}^n \setminus K$ is connected, then every $f \in \mathcal{O}(\mathbb{D}^n \setminus K)$ extends to $F \in \mathcal{O}(\mathbb{D}^n)$ with $F|\mathbb{D}^n \setminus K = f$.*

**Proof**

Every holomorphic function on $\mathbb{D}^n \setminus \rho\mathbb{D}^n$ has a Laurent series expansion.
Since part of the axes intersect the domain, no negative powers occur, hence it is a power series.

# Hartogs' Phenomenon

A *Reinhardt domain* $\Omega \subseteq \mathbb{C}^n$ is such that
$(z_1, \ldots, z_n) \in \Omega \Longrightarrow (\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega$ for all $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.
For $n = 1$, $\Omega$ is an annulus or a disk. In general, $\Omega$ is a union of tori.
Every holomorphic function on $\Omega$ has a unique Laurent series expansion.

**Theorem (Hartogs 1906)**

*Let $K \subset \mathbb{D}^n$ be a compact subset of the open unit polydisk. If $\mathbb{D}^n \setminus K$ is connected, then every $f \in \mathcal{O}(\mathbb{D}^n \setminus K)$ extends to $F \in \mathcal{O}(\mathbb{D}^n)$ with $F|\mathbb{D}^n \setminus K = f$.*

**Proof**

Every holomorphic function on $\mathbb{D}^n \setminus \rho\mathbb{D}^n$ has a Laurent series expansion.
Since part of the axes intersect the domain, no negative powers occur, hence it is a power series.

# Hartogs' Phenomenon

A *Reinhardt domain* $\Omega \subseteq \mathbb{C}^n$ is such that
$(z_1, \dots, z_n) \in \Omega \implies (\lambda_1 z_1, \dots, \lambda_n z_n) \in \Omega$ for all $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.
For $n = 1$, $\Omega$ is an annulus or a disk. In general, $\Omega$ is a union of tori.
Every holomorphic function on $\Omega$ has a unique Laurent series expansion.

**Theorem (Hartogs 1906)**

*Let $K \subset \mathbb{D}^n$ be a compact subset of the open unit polydisk. If $\mathbb{D}^n \setminus K$ is connected, then every $f \in \mathcal{O}(\mathbb{D}^n \setminus K)$ extends to $F \in \mathcal{O}(\mathbb{D}^n)$ with $F|\mathbb{D}^n \setminus K = f$.*

**Proof**

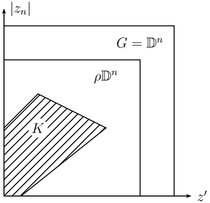
Every holomorphic function on $\mathbb{D}^n \setminus \rho\mathbb{D}^n$ has a Laurent series expansion. Since part of the axes intersect the domain, no negative powers occur, hence it is a power series.

# Hartogs' Phenomenon

A *Reinhardt domain* $\Omega \subseteq \mathbb{C}^n$ is such that
$(z_1, \ldots, z_n) \in \Omega \Longrightarrow (\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega$ for all $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.
For $n = 1$, $\Omega$ is an annulus or a disk. In general, $\Omega$ is a union of tori.
Every holomorphic function on $\Omega$ has a unique Laurent series expansion.

**Theorem (Hartogs 1906)**

*Let $K \subset \mathbb{D}^n$ be a compact subset of the open unit polydisk. If $\mathbb{D}^n \setminus K$ is connected, then every $f \in \mathcal{O}(\mathbb{D}^n \setminus K)$ extends to $F \in \mathcal{O}(\mathbb{D}^n)$ with $F|\mathbb{D}^n \setminus K = f$.*




**Proof**

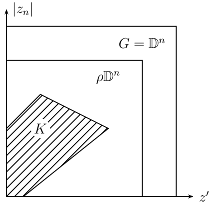
Every holomorphic function on $\mathbb{D}^n \setminus \rho\mathbb{D}^n$ has a Laurent series expansion.
Since part of the axes intersect the domain, no negative powers occur, hence it is a power series.

# Hartogs' Phenomenon

A *Reinhardt domain* $\Omega \subseteq \mathbb{C}^n$ is such that
$(z_1, \ldots, z_n) \in \Omega \Longrightarrow (\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega$ for all $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.
For $n = 1$, $\Omega$ is an annulus or a disk. In general, $\Omega$ is a union of tori.
Every holomorphic function on $\Omega$ has a unique Laurent series expansion.

### Theorem (Hartogs 1906)

*Let $K \subset \mathbb{D}^n$ be a compact subset of the open unit polydisk. If $\mathbb{D}^n \setminus K$ is connected, then every $f \in \mathcal{O}(\mathbb{D}^n \setminus K)$ extends to $F \in \mathcal{O}(\mathbb{D}^n)$ with $F|\mathbb{D}^n \setminus K = f$.*




### Proof

Every holomorphic function on $\mathbb{D}^n \setminus \rho\mathbb{D}^n$ has a Laurent series expansion.
Since part of the axes intersect the domain, no negative powers occur, hence it is a power series.

# Properties of domains

In contrast to the situation in one-variable (Riemann mapping theorem), simply connected domains in $\mathbb{C}^n$ are not all equivalent under holomorphic transformations.

**Theorem (Poincaré 1907)**

*The ball* $\{|z_1|^2 + |z_2|^2 < 1\}$ *and the bidisc* $\{\max\{|z_1|, |z_2|\} < 1\}$ *are not biholomorphically equivalent in* $\mathbb{C}^2$.

This important observation is at the origin of many of the subsequent investigations in Several Complex Variables.

# Properties of domains

One notable property of domains which is analogous in $\mathbb{C}^n$ and $\mathbb{C}$ is the lack of "too many" automorphisms, at least in the bounded case.

**Theorem (H. Cartan 1935)**

*Let $\Omega \subset \mathbb{C}^n$ be a bounded, connected domain. Then any holomorphic automorphism $\psi \colon \Omega \to \Omega$ is determined by its value and its first derivatives at one point $p \in \Omega$ ($1$-jet determination). The set of all automorphisms of $\Omega$ is a finite dimensional Lie group.*

On the other hand, the automorphism group of a generic bounded domain of $\mathbb{C}^n$ ($n \geq 2$) is trivial. Therefore, one needs to look at other invariants in order to try to classify domains.

# Properties of domains

Given a domain $\Omega \subset \mathbb{C}^n$ and a subdomain $U \subset \Omega$, we say that $\widehat{U} \subset \mathbb{C}^n$ containing $U$ is an *analytic completion* of $U$ if, for any $f \in \mathcal{O}(\Omega)$, $f|_U$ extends holomorphically to $\widehat{U}$.

**Definition**

$\Omega$ is called a *domain of holomorphy* if, for any $U \subset \Omega$, all analytic completions $\widehat{U}$ of $U$ are contained in $\Omega$.

Roughly speaking, this means that holomorphic functions in $\Omega$ cannot be all extended past any point of $b\Omega$. In fact, one can show that $\Omega$ is a domain of holomorphy iff it is a domain of existence for some $f \in \mathcal{O}(\Omega)$.

This notion is empty in $\mathbb{C}$, but not in $\mathbb{C}^n$ $(n \geq 2)$ due to the Hartogs' phenomenon. One of the important running threads of SCV has been the characterization and the study of domains of holomorphy.

# Properties of domains

Given a function $f\colon \mathbb{C}^n \to \mathbb{R}$ of class $C^2$, its *Levi form* at $z$ is the hermitian form

$$\mathcal{L}f(\xi, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z)\xi_i \overline{\eta}_j.$$

A domain $\Omega$ of class $C^2$ is *Levi-convex* (or *pseudoconvex*) if the Levi form of any defining function of $b\Omega$, restricted to the tangent space of $b\Omega$, is positive semidefinite.

It was recognized by E.E. Levi in 1910 that pseudoconvexity is a necessary condition for $\Omega$ to be a domain of holomorphy. The question of the converse became known as the *Levi problem*, and was positively settled by Oka in 1950 for domains in $\mathbb{C}^n$. Thus, to understand whether $\Omega$ is a domain of holomorphy it is in principle enough to perform a computation at the boundary.

# CR Geometry

Boundaries of domains need not be biholomorphically equivalent.

**Example**

In $\mathbb{C}^2$, the paraboloid $S = \{\operatorname{Im} w = |z|^2\}$ and the hyperplane $H = \{\operatorname{Im} w = 0\}$ are not equivalent. On $H$, the Levi form is identically zero, while it never vanishes on $S$.

In fact it was Poincaré (1907) who also recognized that there are *infinitely many* equivalence classes of real hypersurfaces under biholomorphic equivalence.

# CR Geometry

In general, higher codimensional submanifolds of $\mathbb{C}^n$ have been studied too. Given $M \subset \mathbb{C}^n$, $p \in M$, the subspace $T_p^c(M) = T_p(M) \cap iT_p(M)$ is called the *complex tangent space* of $M$.

**Definition**

We say that $M$ is a *CR submanifold* if the dimension of $T_p^c(M)$ is constant. This is automatically true in the hypersurface case.

The attempt to classify CR manifolds has been an ongoing program, leading to many advancements in the field and to a complete solution in certain classes.

# CR Geometry

One of the most important results in this line is the local classification of *Levi non-degenerate* hypersurfaces $M$, that is, those for which the Levi form does not admit any vanishing eigenvalue.

**Theorem (Chern–Moser 1975)**

*After applying a holomorphic change of coordinates, a defining function for $M$ can be written in* normal form, *which in principle solves the equivalence problem for hypersurfaces in this class.*

Several differential-geometric invariants arise as a byproduct of the construction in Chern–Moser (such as umbilical points, Levi curvature tensor, chains).

# Mittag-Leffler interpolation

**Theorem (Mittag-Leffler interpolation, 1876, 1884)**

*Let $\Omega \subset \mathbb{C}$ be an open subset and let $A := \{a_j\}_{j \in \mathbb{N}} \subset \Omega$ be a closed discrete subset. Then there exist a meromorphic function $f$ on $\Omega$ such that $f$ has only singularities in $A$ and the principal part $f_j$ of $f$ in each $a_j$ can be prescribed.*

Rephrasing it as an additive Cousin problem:
Every point $a_j$ has a neighborhood $U_j$ with $f_j \in \mathcal{M}(U_j)$ such that $f - f_j \in \mathcal{O}(U_j)$. For each $x \in \Omega \setminus A$ we have a neighborhood $U$ of $x$ where already $f - 0 \in \mathcal{O}(U)$.

**Additive Cousin problem**

An additive Cousin problem on $\Omega \subseteq \mathbb{C}^n$ is given by an open cover $\{U_j\}_{j \in J}$ of $\Omega$ and meromorphic functions $f_j \in \mathcal{M}(U_j)$ s.t. $f_j - f_k \in \mathcal{O}(U_j \cap U_k)$. A solution is $f \in \mathcal{M}(\Omega)$ s.t. $f|U_j - f_j \in \mathcal{O}(U_j)$.

A meromorphic function is locally the quotient of holomorphic functions.

# Mittag-Leffler interpolation

**Theorem (Mittag-Leffler interpolation, 1876, 1884)**

*Let $\Omega \subset \mathbb{C}$ be an open subset and let $A := \{a_j\}_{j\in\mathbb{N}} \subset \Omega$ be a closed discrete subset. Then there exist a meromorphic function $f$ on $\Omega$ such that $f$ has only singularities in $A$ and the principal part $f_j$ of $f$ in each $a_j$ can be prescribed.*

Rephrasing it as an additive Cousin problem:
Every point $a_j$ has a neighborhood $U_j$ with $f_j \in \mathcal{M}(U_j)$ such that $f - f_j \in \mathcal{O}(U_j)$. For each $x \in \Omega \setminus A$ we have a neighborhood $U$ of $x$ where already $f - 0 \in \mathcal{O}(U)$.

**Additive Cousin problem**

An additive Cousin problem on $\Omega \subseteq \mathbb{C}^n$ is given by an open cover $\{U_j\}_{j\in J}$ of $\Omega$ and meromorphic functions $f_j \in \mathcal{M}(U_j)$ s.t. $f_j - f_k \in \mathcal{O}(U_j \cap U_k)$. A solution is $f \in \mathcal{M}(\Omega)$ s.t. $f|U_j - f_j \in \mathcal{O}(U_j)$.

A meromorphic function is locally the quotient of holomorphic functions.

# Mittag-Leffler interpolation

### Theorem (Mittag-Leffler interpolation, 1876, 1884)

*Let $\Omega \subset \mathbb{C}$ be an open subset and let $A := \{a_j\}_{j \in \mathbb{N}} \subset \Omega$ be a closed discrete subset. Then there exist a meromorphic function $f$ on $\Omega$ such that $f$ has only singularities in $A$ and the principal part $f_j$ of $f$ in each $a_j$ can be prescribed.*

Rephrasing it as an additive Cousin problem:
Every point $a_j$ has a neighborhood $U_j$ with $f_j \in \mathcal{M}(U_j)$ such that $f - f_j \in \mathcal{O}(U_j)$. For each $x \in \Omega \setminus A$ we have a neighborhood $U$ of $x$ where already $f - 0 \in \mathcal{O}(U)$.

### Additive Cousin problem

An additive Cousin problem on $\Omega \subseteq \mathbb{C}^n$ is given by an open cover $\{U_j\}_{j \in J}$ of $\Omega$ and meromorphic functions $f_j \in \mathcal{M}(U_j)$ s.t. $f_j - f_k \in \mathcal{O}(U_j \cap U_k)$. A solution is $f \in \mathcal{M}(\Omega)$ s.t. $f|U_j - f_j \in \mathcal{O}(U_j)$.

A meromorphic function is locally the quotient of holomorphic functions.

# Mittag-Leffler interpolation

### Theorem (Mittag-Leffler interpolation, 1876, 1884)

*Let $\Omega \subset \mathbb{C}$ be an open subset and let $A := \{a_j\}_{j\in\mathbb{N}} \subset \Omega$ be a closed discrete subset. Then there exist a meromorphic function $f$ on $\Omega$ such that $f$ has only singularities in $A$ and the principal part $f_j$ of $f$ in each $a_j$ can be prescribed.*

Rephrasing it as an additive Cousin problem:
Every point $a_j$ has a neighborhood $U_j$ with $f_j \in \mathcal{M}(U_j)$ such that $f - f_j \in \mathcal{O}(U_j)$. For each $x \in \Omega \setminus A$ we have a neighborhood $U$ of $x$ where already $f - 0 \in \mathcal{O}(U)$.

### Additive Cousin problem

An additive Cousin problem on $\Omega \subseteq \mathbb{C}^n$ is given by an open cover $\{U_j\}_{j\in J}$ of $\Omega$ and meromorphic functions $f_j \in \mathcal{M}(U_j)$ s.t. $f_j - f_k \in \mathcal{O}(U_j \cap U_k)$. A solution is $f \in \mathcal{M}(\Omega)$ s.t. $f|U_j - f_j \in \mathcal{O}(U_j)$.

A meromorphic function is locally the quotient of holomorphic functions.

# Cousin problems

A meromorphic function is locally the quotient of holomorphic functions.

**Problem (Poincaré)**

*Is every meromorphic function on $\mathbb{C}^n$ the quotient of two holomorphic functions?*

**Theorem (Cousin, 1894)**

*Every Cousin problem in $\mathbb{C}^n$ has a solution (for $n = 2$).*

**Corollary (Cousin, 1894)**

*Every meromorphic function on $\mathbb{C}^2$ is the quotient of two holomorphic functions.*

# Cousin problems

A meromorphic function is locally the quotient of holomorphic functions.

Problem (Poincaré)

*Is every meromorphic function on $\mathbb{C}^n$ the quotient of two holomorphic functions?*

Theorem (Cousin, 1894)

*Every Cousin problem in $\mathbb{C}^n$ has a solution (for $n = 2$).*

Corollary (Cousin, 1894)

*Every meromorphic function on $\mathbb{C}^2$ is the quotient of two holomorphic functions.*

# Cousin problems

A meromorphic function is locally the quotient of holomorphic functions.

**Problem (Poincaré)**

*Is every meromorphic function on $\mathbb{C}^n$ the quotient of two holomorphic functions?*

**Theorem (Cousin, 1894)**

*Every Cousin problem in $\mathbb{C}^n$ has a solution (for $n = 2$).*

**Corollary (Cousin, 1894)**

*Every meromorphic function on $\mathbb{C}^2$ is the quotient of two holomorphic functions.*

# Stein manifolds

## Definition (K. Stein, 1951)

A complex manifold $X$ is called *Stein* if it is holomorphically convex and holomorphically separable.

- $X$ is called holomorphically separable if for any two points $p, q \in X, p \neq q$, there exists $f \in \mathcal{O}(X)$ with $f(p) = 0$ and $f(q) = 1$.
- $X$ is called holomorphically convex if for any compact $K \subset X$ its $\mathcal{O}(X)$-convex hull $\widehat{K}$ is again compact.

## Definition

$$\widehat{K} = \{x \in X \ : \ \forall f \in \mathcal{O}(X) \ |f(x)| \leq \max_K |f|\}$$

## Theorem (Cartan–Thullen 1932)

*A domain $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy.*

# Stein manifolds

### Definition (K. Stein, 1951)

A complex manifold $X$ is called *Stein* if it is holomorphically convex and holomorphically separable.

- $X$ is called holomorphically separable if for any two points $p, q \in X, p \neq q$, there exists $f \in \mathcal{O}(X)$ with $f(p) = 0$ and $f(q) = 1$.
- $X$ is called holomorphically convex if for any compact $K \subset X$ its $\mathcal{O}(X)$-convex hull $\widehat{K}$ is again compact.

### Definition

$$\widehat{K} = \{x \in X \ : \ \forall f \in \mathcal{O}(X) \ |f(x)| \leq \max_K |f|\}$$

### Theorem (Cartan–Thullen 1932)

*A domain $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy.*

# Stein manifolds

**Definition (K. Stein, 1951)**

A complex manifold $X$ is called *Stein* if it is holomorphically convex and holomorphically separable.

- $X$ is called holomorphically separable if for any two points $p, q \in X, p \neq q$, there exists $f \in \mathcal{O}(X)$ with $f(p) = 0$ and $f(q) = 1$.
- $X$ is called holomorphically convex if for any compact $K \subset X$ its $\mathcal{O}(X)$-convex hull $\widehat{K}$ is again compact.

**Definition**

$$\widehat{K} = \{x \in X \ : \ \forall f \in \mathcal{O}(X)\ |f(x)| \leq \max_K |f|\}$$

**Theorem (Cartan–Thullen 1932)**

*A domain $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy.*

# Stein manifolds

**Definition (K. Stein, 1951)**

A complex manifold $X$ is called *Stein* if it is holomorphically convex and holomorphically separable.

- $X$ is called holomorphically separable if for any two points $p, q \in X, p \neq q$, there exists $f \in \mathcal{O}(X)$ with $f(p) = 0$ and $f(q) = 1$.
- $X$ is called holomorphically convex if for any compact $K \subset X$ its $\mathcal{O}(X)$-convex hull $\widehat{K}$ is again compact.

**Definition**

$$\widehat{K} = \{x \in X \ : \ \forall f \in \mathcal{O}(X) \ |f(x)| \leq \max_K |f|\}$$

**Theorem (Cartan–Thullen 1932)**

*A domain $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy.*

# Cartan's Theorems

**Theorem (Theorem B, H. Cartan 1951, Serre 1953)**

*Let $\mathcal{S}$ be a coherent sheaf on a Stein manifold $X$. Then*

$$H^p(X, \mathcal{S}) = 0 \quad \text{for } p \geq 1$$

**Example**

- holomorphic functions $\mathcal{O}$ (Oka 1950)
- holomorphic functions $\mathcal{O}_Y$ vanishing on a subvariety $Y$
- holomorphic vector bundles

Every short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to$$

In particular, $H^1(X, \mathcal{F}) = 0$ implies that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

# Cartan's Theorems

**Theorem (Theorem B, H. Cartan 1951, Serre 1953)**

*Let $\mathcal{S}$ be a coherent sheaf on a Stein manifold $X$. Then*

$$H^p(X, \mathcal{S}) = 0 \quad \text{for } p \geq 1$$

**Example**

- holomorphic functions $\mathcal{O}$ (Oka 1950)
- holomorphic functions $\mathcal{O}_Y$ vanishing on a subvariety $Y$
- holomorphic vector bundles

Every short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to$$

In particular, $H^1(X, \mathcal{F}) = 0$ implies that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

# Cartan's Theorems

Theorem (Theorem B, H. Cartan 1951, Serre 1953)

*Let $\mathcal{S}$ be a coherent sheaf on a Stein manifold $X$. Then*

$$H^p(X, \mathcal{S}) = 0 \quad \text{for } p \geq 1$$

Example

- holomorphic functions $\mathcal{O}$ (Oka 1950)
- holomorphic functions $\mathcal{O}_Y$ vanishing on a subvariety $Y$
- holomorphic vector bundles

Every short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to$$

In particular, $H^1(X, \mathcal{F}) = 0$ implies that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

# Properties of Stein manifolds

**Theorem**

*Every additive Cousin problem on a Stein manifold has a solution.*

Proof.

$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$ leads to

$$0 \to \mathcal{O}(X) \to \mathcal{M}(X) \to \mathcal{M}/\mathcal{O}(X) \to H^1(X, \mathcal{O}) = 0$$

□

**Theorem (Weil 1935, Oka 1937, Stein 1951)**

*Let $X$ be a Stein manifold. Let $K \subset X$ with $K = \widehat{K}$. Then every function holomorphic in a neighborhood of $K$ can be approximated by $F \in \mathcal{O}(X)$ uniformly on $K$.*

**Theorem (Remmert 1956)**

*Every Stein manifold can be properly embedded in some $\mathbb{C}^N$.*

# Properties of Stein manifolds

**Theorem**

*Every additive Cousin problem on a Stein manifold has a solution.*

**Proof.**

$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$ leads to

$$0 \to \mathcal{O}(X) \to \mathcal{M}(X) \to \mathcal{M}/\mathcal{O}(X) \to H^1(X, \mathcal{O}) = 0$$

□

**Theorem (Weil 1935, Oka 1937, Stein 1951)**

*Let $X$ be a Stein manifold. Let $K \subset X$ with $K = \widehat{K}$. Then every function holomorphic in a neighborhood of $K$ can be approximated by $F \in \mathcal{O}(X)$ uniformly on $K$.*

**Theorem (Remmert 1956)**

*Every Stein manifold can be properly embedded in some $\mathbb{C}^N$.*

# Properties of Stein manifolds

**Theorem**

*Every additive Cousin problem on a Stein manifold has a solution.*

**Proof.**

$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$ leads to

$$0 \to \mathcal{O}(X) \to \mathcal{M}(X) \to \mathcal{M}/\mathcal{O}(X) \to H^1(X, \mathcal{O}) = 0$$

□

**Theorem (Weil 1935, Oka 1937, Stein 1951)**

*Let $X$ be a Stein manifold. Let $K \subset X$ with $K = \widehat{K}$. Then every function holomorphic in a neighborhood of $K$ can be approximated by $F \in \mathcal{O}(X)$ uniformly on $K$.*

**Theorem (Remmert 1956)**

*Every Stein manifold can be properly embedded in some $\mathbb{C}^N$.*

# Properties of Stein manifolds

### Theorem

*Every additive Cousin problem on a Stein manifold has a solution.*

### Proof.

$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$ leads to

$$0 \to \mathcal{O}(X) \to \mathcal{M}(X) \to \mathcal{M}/\mathcal{O}(X) \to H^1(X, \mathcal{O}) = 0$$

□

### Theorem (Weil 1935, Oka 1937, Stein 1951)

*Let $X$ be a Stein manifold. Let $K \subset X$ with $K = \widehat{K}$. Then every function holomorphic in a neighborhood of $K$ can be approximated by $F \in \mathcal{O}(X)$ uniformly on $K$.*

### Theorem (Remmert 1956)

*Every Stein manifold can be properly embedded in some $\mathbb{C}^N$.*

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

**Definition**

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f : Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t : Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

**Definition**

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f : Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t : Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

### Definition

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f : Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t : Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

**Definition**

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f\colon Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t\colon Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

### Definition

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f\colon Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t\colon Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0,1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

### Definition

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

- Let $Y$ be a Stein manifold and let $f\colon Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t\colon Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

# Oka manifolds

- $X$ Stein: many $X \to \mathbb{C}^k$
- $X$ Oka: many $\mathbb{C}^k \to X$

**Definition**

A complex manifold $X$ is called *Oka* if is satisfies the *Basic Oka Principle with Interpolation*:

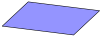
- Let $Y$ be a Stein manifold and let $f\colon Y \to X$ be a continuous map. There exists a homotopy of continuous maps $f_t\colon Y \to X$ such that $f_0 = f$ and $f_1$ is holomorphic.
- Let $A \subset Y$ be an analytic subset and let $f$ be holomorphic in a neighborhood of $A$. Then we can choose the homotopy $f_t|A = f|A$ for all $t \in [0, 1]$.

An analytic subset is a closed subset which is locally the common zero locus of some holomorphic functions.

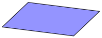
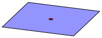
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka | | |
| not Oka | | |

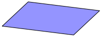
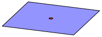
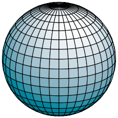
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$ | |
| not Oka | | |

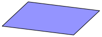
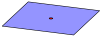
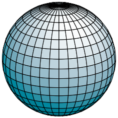
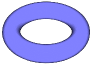
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ | |
| not Oka | | |

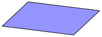
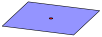
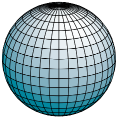
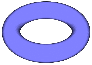
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$ |
| not Oka | | |

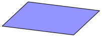
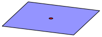
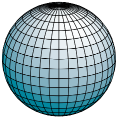
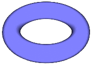
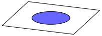
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka | | |

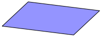
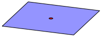
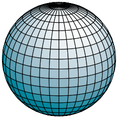
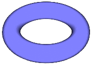
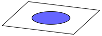
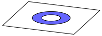
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka | all other non-compact | |

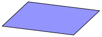
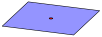
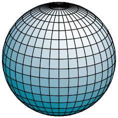
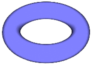
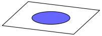
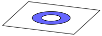
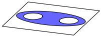
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |  all other non-compact | |

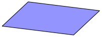
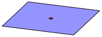
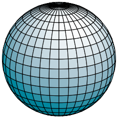
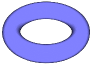
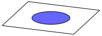
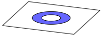
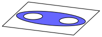
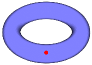
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |   all other non-compact | |

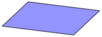
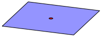
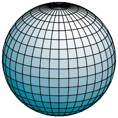
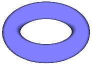
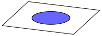
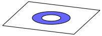
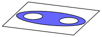
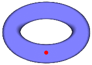
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |    all other non-compact | |

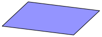
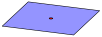
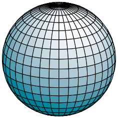
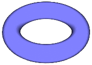
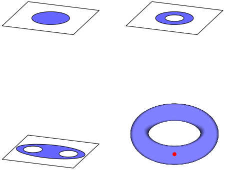
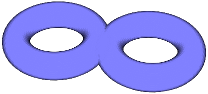
# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |     all other non-compact | |

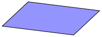
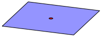
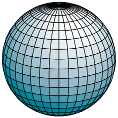
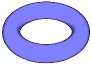
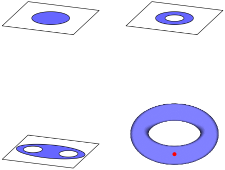
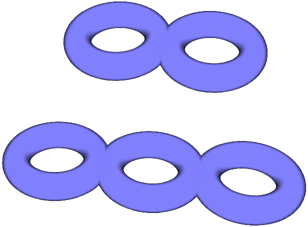
# Examples: Riemann surfaces

| | Stein | not Stein |
| --- | --- | --- |
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |     all other non-compact | all other compact |

# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |  all other non-compact |  all other compact |

# Examples: Riemann surfaces

| | Stein | not Stein |
|---|---|---|
| Oka |  $\mathbb{C}$  $\mathbb{C}^*$ |  $\mathbb{CP}^1$  tori |
| not Oka |  all other non-compact |  all other compact |

# Elliptic manifolds

Gromov introduced the notion of an *elliptic* complex manifold using *dominating sprays*. This condition is much easier to verify.

**Theorem (Gromov 1989, Forstnerič–Prezelj 2000)**

*An elliptic complex manifold is Oka.*

**Example**

1. complex-affine space $\mathbb{C}^n$
2. Complex Lie groups
3. homogeneous spaces of complex Lie groups
4. complex-projective space $\mathbb{CP}^n$
5. direct products of Oka manifolds

# Elliptic manifolds

Gromov introduced the notion of an *elliptic* complex manifold using *dominating sprays*. This condition is much easier to verify.

Theorem (Gromov 1989, Forstnerič–Prezelj 2000)

*An elliptic complex manifold is Oka.*

Example

1. complex-affine space $\mathbb{C}^n$
2. Complex Lie groups
3. homogeneous spaces of complex Lie groups
4. complex-projective space $\mathbb{CP}^n$
5. direct products of Oka manifolds

# Elliptic manifolds

Gromov introduced the notion of an *elliptic* complex manifold using *dominating sprays*. This condition is much easier to verify.

**Theorem (Gromov 1989, Forstnerič–Prezelj 2000)**

*An elliptic complex manifold is Oka.*

**Example**

1. complex-affine space $\mathbb{C}^n$
2. Complex Lie groups
3. homogeneous spaces of complex Lie groups
4. complex-projective space $\mathbb{CP}^n$
5. direct products of Oka manifolds

# Elliptic manifolds

Gromov introduced the notion of an *elliptic* complex manifold using *dominating sprays*. This condition is much easier to verify.

**Theorem (Gromov 1989, Forstnerič–Prezelj 2000)**

*An elliptic complex manifold is Oka.*

**Example**

1. complex-affine space $\mathbb{C}^n$
2. Complex Lie groups
3. homogeneous spaces of complex Lie groups
4. complex-projective space $\mathbb{CP}^n$
5. direct products of Oka manifolds

# Elliptic manifolds

Gromov introduced the notion of an *elliptic* complex manifold using *dominating sprays*. This condition is much easier to verify.

**Theorem (Gromov 1989, Forstnerič–Prezelj 2000)**

*An elliptic complex manifold is Oka.*

**Example**

1. complex-affine space $\mathbb{C}^n$
2. Complex Lie groups
3. homogeneous spaces of complex Lie groups
4. complex-projective space $\mathbb{CP}^n$
5. direct products of Oka manifolds

# Partial differential equations

The antiholomorphic differential $\overline{\partial} f = \sum_j \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$ can be naturally extended to an operator between forms

$$\overline{\partial}\colon \Omega^{(r,s)}(U) \to \Omega^{(r,s+1)}(U).$$

As it turns out, the study of the $\overline{\partial}$-equation $\overline{\partial}\alpha = \beta$ is deeply interconnected with the study of the complex geometric properties of $U \subset \mathbb{C}^n$.

Many classical problems can be reformulated in terms of the solvability of $\overline{\partial}$, such as the Cousin problems and the existence of a global defining equation for a complex submanifold.

# Partial differential equations

The study of the $\overline{\partial}$-equation is often carried out in the space of functions/forms on $U \subset \mathbb{C}^n$ which are in $L^2$ with respect to the Lebesgue measure, or certain special weighted measures (cf. Hörmander 1966)

The space $L^2\mathcal{O}(U)$ of $L^2$ holomorphic functions on $U$ is called the *Bergman space*. Because of the Cauchy formula, the evaluation functionals

$$L^2\mathcal{O}(U) \ni f \to f(z) \in \mathbb{C}$$

are continuous.

The kernel $K(z, \cdot)$ representing these functionals, called the *Bergman kernel* (Bergman 1922), has been extensively studied (see e.g. Fefferman 1974) and used to define an invariant metric on the domain. Its behaviour is linked to the properties of the boundary of $U$.

# Partial differential equations

One of the fundamental results of the theory is the solvability of $\overline{\partial}$ on pseudoconvex domains:

### Theorem

*Let $U$ be a pseudoconvex domain, and $\beta$ be a $\overline{\partial}$-closed smooth form on $U$. Then there exists a smooth form $\alpha$ on $U$ such that $\overline{\partial}\alpha = \beta$.*

One of the consequences of the solvability of $\overline{\partial}$ is the equivalence of domain of holomorphy and pseudoconvex domains (the Levi problem mentioned earlier).
The Dolbeault Lemma says that the sheaf cohomology of $\mathcal{O}$ is the cohomology of the following sequence:

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,2} \xrightarrow{\overline{\partial}} \dots$$

hence $H^{q,p}_{\overline{\partial}}(X) \cong H^p(X, \mathcal{O})$ which vanishes by Cartan's Theorem B if $X$ is Stein.