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Harsh Default Penalties Lead to Ponzi Schemes

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Abstract: In the presence of utility penalties, collateral requirements do not always eliminate the occurrence of Ponzi schemes. Harsh utility penalties may induce effective payments over collateral recollection values. In this event, loans can be larger than collateral costs and Ponzi schemes become possible.

KEYWORDS: Equilibrium, Incomplete markets, Default, Collateral, Utility Penalties, Ponzi Schemes.
JEL CLASSIFICATION: D52, D91.

1 Introduction.

In pioneering works published thirty years ago, Martin Shubik initiated the general equilibrium theory of default. Shubik (1972) studied an optimal bankruptcy rule and its implications for Pareto optimality. Shubik and Wilson (1977) introduced for the first time a linear and separable default penalty in the utility function. Dubey and Shubik (1979) used the same penalty function to discuss optimal bankruptcy rules. A decade later, Dubey et al. (1990) started introducing default in the general equilibrium model of incomplete markets. This program culminated in a very recent comprehensive publication (Dubey et al. (2005)) and has influenced the work of many others. When default is allowed (but penalized in the utility function) economic efficiency increases (Zame (1993)) and existence of equilibria becomes compatible with a continuum of states (Araujo et al. (1996, 1998)). Under another form of default punishment, which is the seizure of a collateral, equilibrium exists even in the case of real assets (Geanakoplos (1997) and Dubey et al. (1995)).

The current paper resumes, in the infinite horizon setting, an issue raised by Martin Shubik in his path breaking works: how harsh should be the default enforcement mechanism ? Collateral was shown to avoid Ponzi schemes (Araujo et al. (2002)) and to rationalize tight borrowing limits in computational stationary equilibria (Kubler and Schmedders (2003)). We claim that if, in addition to the seizure of the collateral, an infinite lived agent faces utility penalties (of the form proposed by Shubik and Wilson (1977)), that are too harsh, then the agent may prefer a Ponzi scheme to the default strategy of surrendering the collateral when it no longer covers the debt. That is, under harsh default penalties equilibrium may not exist.

Utility penalties can be interpreted as social sanctions or loss of reputation and constitute an important mechanism to encourage agents to pay back their loans. The other mechanism is to require borrowers to secure their short sales using durable goods or assets as collateral. When commodities serve as collateral, short sales become endogenously bounded and this suffices to guarantee existence of equilibrium in finite horizon economies. However, when the horizon is infinite, agents may use new credit to pay back previous loans in a sequentially increasing way. These Ponzi schemes may occur even at non-arbitrage prices and the decision problem of an agent may not have a solution. In the literature of incomplete markets without default, these Ponzi schemes have been avoided by imposing borrowing constraints node by node or a transversality condition that limits the rate of growth of debts (e.g., Kehoe and Levine (1993), Magill and Quinzii (1994, 1996) , Hernandez and Santos (1996)). When default is allowed and the seizure of the collateral is the only enforcement mechanism, Ponzi schemes can be ruled out without having to impose borrowing constraints or transversality conditions (see Araujo et al. (2002)).

However, when utility penalties are also present Ponzi schemes may reappear. Combining short sales with the purchase of a collateral constitutes a joint operation that yields nonnegative returns in the absence of utility penalties (as effective payments never exceed the value of the depreciated collateral). By non-arbitrage, this joint operation should have a nonnegative net price, implying that collateral costs must exceed loans and, therefore, the real value of what is borrowed becomes uniformly bounded. But when utility penalties are also present borrowers can have an incentive to pay back more than the value of the depreciated collateral and the above joint operation will no longer have nonnegative returns and nonnegative net prices, that is the collateral cost may now be lower than the loan. We show that in such an event Ponzi schemes are possible. In fact, short sales are still bounded node by node (due to the obligation of constituting collateral using durable goods of limited endowments) but the real value of what may be borrowed is no longer bounded. Hence, if penalties are too harsh the decision problem of infinite lived agents may fail to have an optimal solution.

Under moderate penalties, equilibrium exists and we can actually refine it to guarantee that it is non–trivial in the sense that when assets are not traded the market payment rate (the average of the individual effective repayment rates) can be set different from zero. A pure spot markets equilibrium with zero asset prices and zero payment rates may be regarded as a pessimistic outcome where financial trades do not occur due to beliefs that everyone will default entirely. It is also a trivial outcome since it always exists, just like in the finite horizon case (as the transversality condition on consumption plans is trivially satisfied). In models where assets are not backed by collateral,

utility penalties have to be high enough to allow for non–trivial equilibria (see Dubey et al (2005)). In collateralized economies this is not the case fortunately: if penalties are low enough, Ponzi schemes are avoided and equilibria are actually non-trivial.

The contents of the paper are as follows. We discuss first the possibility of doing Ponzi schemes in the presence of default penalties. Then, we establish existence of equilibrium under an assumption that bounds utility penalties associated with maximal default (when defaulters pay back just the value of the depreciated collateral). Notice that we may depart from the classical interiority assumption on endowments by allowing for null endowments of durable goods beyond the initial date. Existence is obtained by proving individual optimality of cluster points of truncated finite horizon equilibria. Finally, we give examples illustrating how harsh penalties may generate Ponzi schemes. Indeed, we show that to allow for Ponzi schemes, penalties do not need to be high enough so that default becomes prohibited, but it is sufficient that penalties ensure that default will not be maximal.

2 The Model.

Stochastic Structure.

We consider a discrete time economy with infinite horizon and uncertainty. The stochastic structure of this model is described by an infinite tree with an unique root and finitely many branches at each node. Formally, let $\mathcal{T} = \{0, 1, \cdots\}$ be the set of dates and let $\mathbf{F}_t$ be the finite set of histories that may occur up to time t.

A pair $\xi = (t, \sigma)$ where $t \in \mathcal{T}$ and $\sigma \in F_t$ is called node and $t(\xi) = t$ is the date of node ξ. The set D consisting of all nodes is called the event-tree induced by $\mathbf{F}$:

$$D = \bigcup_{\substack{t \in \mathcal{T} \\ \sigma \in F_t}} \{(t, \sigma)\}.$$

A node $\xi' = (t', \sigma')$ is said to succeed (resp. strictly) node $\xi = (t, \sigma)$ if $t' \geq t$ (resp. $t' > t$) and $\sigma' \subset \sigma$. We write $\xi' \geq \xi$ (resp. $\xi' > \xi$). Let $\xi \in D$. We will denote by:

- $D(\xi)$ the subtree of the nodes which succeed ξ,
- $D^+(\xi) = \{\xi' \in D | \xi' > \xi\}$ the set of the strict successors of ξ,
- $D_T(\xi)$ the subset of nodes of $D(\xi)$ at date T,
- $D^T(\xi)$ the subset of nodes of $D(\xi)$ between $t(\xi)$ and T,

- $\xi^+ = \{\eta \in D(\xi) | t(\eta) = t(\xi)+1\}$ the set of immediate successors of ξ. The number of elements of ξ^+, called the branching number, is assumed to be finite.

If $\xi = (t, \sigma),\ t \geq 1$, the unique node $\xi^- = (t-1, \sigma'),\ \sigma \subset \sigma'$ is called the predecessor of ξ.
When ξ is the initial node, denoted ξ_0, the notations are simplified to $D^+,\ D_T,\ D^T$.

Commodity, Financial and Demographic Structures.
At each node $\xi \in D$, a finite number G of physical goods, indexed by $g = 1, \ldots, G$, are traded on spot markets. These goods may be durable and suffer a partial depreciation from a period to another. The structure of depreciation in the event-tree is given by a collection of $G \times G-$matrices $Y := \{Y(\xi)\}_{\xi \in D}$. As in Araujo et al (2002), we assume that $Y(\xi)$ is a diagonal matrix, $\big(\mathrm{diag}(a(\xi, g))\big)$, for each node $\xi \in D$. A commodity $g \in G$ is durable at node $\xi \in D$ if $a(\xi, g)$ is different from zero and perishes at ξ otherwise.
Denote by $p(\xi) = (p(\xi, g),\ g \in G)$ the vector of spot prices for the G goods at node ξ and by $p = (p(\xi),\ \xi \in D) \in \mathbb{R}_+^{D \times G}$ the spot price process.
At each node of the event-tree, there is a set $J(\xi)$ consisting of a finite number $\iota(\xi)$ of one-period real assets, available for intertemporal transaction and insurance. Let $A^j(\xi) \in \mathbb{R}_+^G \setminus \{0\}$ be the return, at node ξ, in quantities of the G goods, of one unit of the asset $j \in J(\xi^-)$. We denote $A(\xi) = (A^j(\xi))_{j \in J(\xi^-)}$, and $A := \prod_{\xi \in D} A(\xi)$. Given commodity prices $p(\xi)$, the vector $p(\xi)A^j(\xi)$ expresses the financial return, denominated in units of account, of one unit of asset j. Thus, at each node $\xi \in D$, the $(\#\xi^+ \times \iota(\xi))-$matrix

$$V(p) := (p(\eta)A^j(\eta))_{\substack{\eta \in \xi^+ \\ j \in J(\xi)}}$$

completely describes the financial promise at time $t(\xi)+1$ allowed by the real asset structure. Let $q(\xi) = (q(\xi, j),\ j \in J(\xi)) \in \mathbb{R}_+^{\iota(\xi)}$ be the vector of prices of the securities issued at node ξ and let $q = (q(\xi), \xi \in D)$ denote the security price process which belongs to security price space $\prod_{\xi \in D} \mathbb{R}_+^{\iota(\xi)}$.
The demographic structure of the model is given by a finite set I of infinitely-lived agents. The cardinality of I will be denoted by $\mathcal{I}$. At each node ξ, an agent $i \in I$ has a portfolio $z^i(\xi) := (z^i_j(\xi),\ j \in J(\xi))$, with $z^i(\xi) = \theta^i(\xi) - \varphi^i(\xi)$ where:

- $\theta^i(\xi) := (\theta^i_j(\xi),\ j \in J(\xi)) \in \mathbb{R}_+^{\iota(\xi)}$ are the quantities of assets bought by agent i at node ξ,
- $\varphi^i(\xi) := (\varphi^i_j(\xi),\ j \in J(\xi)) \in \mathbb{R}_+^{\iota(\xi)}$ is the short–sale of assets by i at node ξ.

Default Penalties and Collateral.
As in Dubey et al. (1995) and Araujo et al. (2002), the short sale of each unit of asset $j \in J(\xi)$ must be secured by durable goods, according to exogenously given collateral coefficients $C^j(\xi) := (C^j_g(\xi),\ g \in G) \in \mathbb{R}_+^G \setminus \{0\}$. We denote $C := (C^j(\xi), \xi \in D,\ j \in J(\xi))$. Let $\tilde{x}(\xi) = C(\xi)\,\varphi(\xi) + x(\xi)$ be the total consumption bundle at node ξ, where the first term denotes the collateral bundle and the second one the consumption in excess of the collateral. The collateral requirements imply a nonnegativity constraint on the latter, $x(\xi) \geq 0$.
At each node $\xi \in D$, the debt of an agent $i \in I$ (induced by the sale of an asset $j \in J(\xi^-)$) is $p(\xi)A^j(\xi)\varphi^i_j(\xi^-)$ and the effective payment (in units of account) is $\Delta^i_j(\xi)$. A default $[p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \Delta^i_j(\xi)] > 0$ implies the seizure of the collateral by the creditors and gives the borrowers a disutility represented by separable and linear penalty functions, as in Dubey et al. (2005) pioneering work on default. The penalty depends actually on real default: at node $\xi \in D$, for a fixed bundle $b(\xi) = (b(\xi, g), g \in G) \in \mathbb{R}_{++}^G$, consumer i's penalty rate for asset $j \in J(\xi^-)$ is $\lambda^i_j(\xi)/p(\xi)\,b(\xi)$, and is well defined for $p(\xi) > 0$. More precisely, the preferences of each agent $i \in I$ are represented by an additively time-node separable utility function U^i defined as follows: for a price process $p(\xi) > 0$, at any node ξ,

$$U^i(\tilde{x}^i, \theta^i, \varphi^i, \Delta^i) := \sum_{\xi \in D} v^i_\xi(\tilde{x}^i(\xi)) - \sum_{\xi \in D \setminus \{0\}} \sum_{j \in J(\xi^-)} \lambda^i_j(\xi) \frac{[p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \Delta^i_j(\xi)]^+}{p(\xi)b(\xi)}, \tag{1}$$

where $\forall i \in I,\ \forall \xi \in D,\ v^i_\xi : \mathbb{R}_+^G \longrightarrow \mathbb{R}$. At each node ξ, $[p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \Delta^i_j(\xi)]^+ = \max\{p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \Delta^i_j(\xi), 0\}$ is the default of the agent $i \in I$ on his promise for the sale of the asset j at node ξ^-. In the absence of default penalties, each seller of asset $j \in J(\xi)$ would deliver at each immediate successor $\eta \in \xi^+$, exactly the minimum between the value of his debts and the value of the depreciated collateral. In the presence of default penalties, defaulters will deliver *at least* the value of the depreciated collateral but may choose to deliver more than this. That is,

$$\min\{p(\eta)A^j(\eta), p(\eta)Y(\eta)C^j(\xi)\}\ \varphi^i_j(\xi) \leq \Delta^i_j(\eta) \leq p(\eta)A^j(\eta)\varphi^i_j(\xi).$$

Consequently, unlike in Araujo et al. (2002), different agents tend to choose different effective payment rates. This was already the case in Dubey et al. (1995). On the other side, in view of the anonymity of the markets, lenders do not know what will be the payments of each individual borrower. Therefore, we need to introduce variables representing the expected deliveries of the sellers. Let $(K^j(\xi) \in [0,1],\ \xi \in D,\ j \in J(\xi^-))$ be *the expected delivery rates on asset* j at node ξ. This variable is taken as given by the agents and will be determined endogenously at equilibrium, as in Dubey et al. (2005).

We define the Economy $\mathcal{E}$ as follows: $\mathcal{E} := ((\omega^i, \lambda^i, U^i)_{i\in I}, A, C, Y)$.
Let us define the budget sets of the agents:

Definition 2.1 *[Budget sets]*
Given (p,q,K), the budget set $B^i(p,q,K)$ of an agent $i \in I$ is the set of $(x^i, \theta^i, \varphi^i, \Delta^i)$ in $\mathbb{R}_+^{G\times D} \times \prod_{\xi\in D} \mathbb{R}_+^{\iota(\xi)} \times \prod_{\xi\in D} \mathbb{R}_+^{\iota(\xi)} \times \prod_{\xi\in D} \mathbb{R}_+^{\iota(\xi)\times G}$ verifying:

$$p(\xi_0)\cdot(x^i(\xi_0) - \omega^i(\xi_0)) + p(\xi_0)C(\xi_0)\varphi^i(\xi_0) + q(\xi_0)\cdot\big(\theta^i(\xi_0) - \varphi^i(\xi_0)\big) \le 0, \qquad (2)$$

and $\forall \xi \in D\setminus\{\xi_0\}$,

$$p(\xi)\cdot(x^i(\xi) - \omega^i(\xi)) + p(\xi)C(\xi)\varphi^i(\xi) + q(\xi)\cdot\big(\theta^i(\xi) - \varphi^i(\xi)\big) + \sum_{j\in J(\xi^-)} \Delta^i_j(\xi)$$
$$\le p(\xi)[Y(\xi)x^i(\xi^-) + Y(\xi)C(\xi^-)\varphi^i(\xi^-)] + \sum_{j\in J(\xi^-)} K^j(\xi)p(\xi)A^j(\xi)\theta^i_j(\xi^-), \qquad (3)$$

$$\Delta^i_j(\xi) \ge \varphi^i_j(\xi^-)\ \min\{p(\xi)A^j(\xi), p(\xi)Y(\xi)C^j(\xi)\}. \qquad (4)$$

Let us adopt the following normalization: for each node $\xi \in D$, $\|p(\xi)\|_1 + \|q(\xi)\|_1 = 1$.
We denote by $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ : \|x\|_1 = 1\}$.
Now, we are ready to define the equilibrium of our model:

Definition 2.2 *[Equilibrium]*
An equilibrium of $\mathcal{E}$ is a vector $(\overline{p}, \overline{q}, \overline{K}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i\in I})$ such that $\overline{p}(\xi) > 0$ at any node $\xi \in D$ and verifying:

(i) *For each agent $i \in I$,* $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) \in \operatorname{Argmax} U^i(x,\theta,\varphi,\Delta)$ *over* $B^i(\overline{p}, \overline{q}, \overline{K})$,

(ii) $\sum_{i\in I}[\overline{x}^i(\xi_0) + C(\xi_0)\overline{\varphi}^i(\xi_0)] = \sum_{i\in I}\omega^i(\xi_0)$,

(iii) $\sum_{i\in I}[\overline{x}^i(\xi) + C(\xi)\overline{\varphi}^i(\xi)] = \sum_{i\in I}[\omega^i(\xi) + Y(\xi)\overline{x}^i(\xi^-) + Y(\xi)C(\xi^-)\overline{\varphi}^i(\xi^-)],\ \forall \xi \in D\setminus\{0\}$,

(iv) $\sum_{i\in I}\overline{\theta}^i = \sum_{i\in I}\overline{\varphi}^i$,

(v) $\forall \xi \in D\setminus\{\xi_0\},\ \forall j \in J(\xi^-),\ \overline{K}^j(\xi)\sum_{i\in I}\overline{p}(\xi)A^j(\xi)\overline{\theta}^i_j(\xi^-) = \sum_{i\in I}\overline{\Delta}^i_j(\xi)$.

Condition (i) is the optimality of the agents choices over their budget sets. Conditions (ii), (iii) and (iv) require the commodity and asset markets to clear. Condition (v) says that, at each node and for each asset, the total expected delivery to the buyers (lenders) is equal to the total effective delivery made by the sellers (borrowers).

Remark 2.1 Even though returns from asset purchases are endogenous (since borrowers have personalized repayment rates, due to the presence of utility penalties), the collateral structure allows us to refine the equilibrium concept to make it non–trivial in the sense that, in the absence of asset trades, the repayment rates K^j can be taken to be nonnull. In fact, we will show that these rates can be set greater or equal to $\frac{\min\{p(\xi)A^j(\xi),\, p(\xi)Y(\xi)C^j(\xi^-)\}}{p(\xi)A^j(\xi)}$ (where p is different from zero at equilibrium) which is bounded from below by $\min\{1, \frac{y(\xi)\, c^j(\xi^-)\, \delta_j(\xi)}{\overline{A}^j(\xi)}\}$, with: $y(\xi) = \min\{a(\xi,g) : a(\xi,g) > 0\}$, $c^j(\xi) = \min\{C^j_g(\xi) : C^j_g(\xi) > 0\}$, $\overline{A}^j(\xi) = \max\{A^j_g(\xi), g \in G\}$ and $\delta_j(\xi)$ is a positive lower bound for $\sum_{g\in S(\xi,j)} p(\xi,g)$, where $S(\xi,j) = \{g \in G : a(\xi,g) > 0 \text{ and } C^j_g(\xi) > 0\}$.
The lower bound $\delta_j(\xi)$ will be given explicitly later on by Inequality (21) of Corollary 5.1. Hence, the model does not suffer from the possibility that asset trades are zero due to pessimistic beliefs about the endogenous returns, as could happen in the absence of collateral and for low utility penalties (See Dubey et al. (2005)). This motivates the following definition.

Definition 2.3 *[Non–trivial equilibrium.]*
A non–trivial equilibrium $(\overline{p}, \overline{q}, \overline{K}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i\in I})$ of $\mathcal{E}$ is an equilibrium such that for any (ξ, j), we have $(\overline{\theta}_j(\xi), \overline{\varphi}_j(\xi))$ different from zero or $\overline{K}^j(\xi) > 0$.

3 Opportunity of doing Ponzi schemes.

In this section, we explain why Ponzi schemes may exist when default penalties are introduced in an infinite horizon incomplete market model with exogenous collateral requirement. Suppose the price process (p,q) is such that one can find $\tilde{\varphi}$ satisfying at a node ξ and its successors the following inequalities, for some $j_\xi \in J(\xi)$, $j_\sigma \in J(\sigma)$,

$$[p(\xi)C^{j_\xi}(\xi) - q^{j_\xi}(\xi)]\hat{\varphi}^i_{j_\xi}(\xi) < 0. \qquad (5)$$

and

$$[p(\sigma)C^{j_\sigma}(\sigma) - q^{j_\sigma}(\sigma)]\hat{\varphi}^i_{j_\sigma}(\sigma) < p(\sigma)[Y(\sigma)C^{j_{\sigma^-}}(\sigma^-) - A^{j_{\sigma^-}}(\sigma)]\hat{\varphi}^i_{j_{\sigma^-}}(\sigma^-),\ \forall \sigma \in D^+(\xi). \tag{6}$$

That is, the joint operation of constituting collateral and short selling (net of the returns from this same joint operation at the preceding node) always yields some income. In such an event, the agent can improve upon any budget feasible vectors of short–sales φ^i and effective payments Δ^i by taking new vectors $\tilde{\varphi}^i$ and $\tilde{\Delta}^i$, given as follows:

$$\forall \sigma \in D(\xi),\ \forall j \in J(\sigma),\ \tilde{\varphi}^i_j(\sigma) = \begin{cases} \varphi^i_{j_\sigma}(\sigma) + \hat{\varphi}^i_{j_\sigma}(\sigma), & \text{if } j = j_\sigma, \\ \varphi^i_j(\sigma), & \text{if } j \neq j_\sigma, \end{cases} \text{, for some } j_\sigma \in J(\sigma),$$

$$\forall \sigma \in D^+(\xi),\ \forall j \in J(\sigma^-),\ \tilde{\Delta}^i_j(\sigma) = \begin{cases} \Delta^i_{j_{\sigma^-}}(\sigma) + p(\sigma)A^{j_{\sigma^-}}(\sigma)\hat{\varphi}^i_{j_{\sigma^-}}(\sigma^-) & \text{if } j = j_{\sigma^-} \\ \Delta^i_j(\sigma) & \text{if } j \neq j_{\sigma^-} \end{cases},$$

for $j_{\sigma^-} \in J(\sigma^-)$.

That is, from node ξ onwards, the short–sale of some asset is increased (but this asset may change from node to node) and the effective deliveries on this additional short–sale is equal to the promised delivery.

Then, $(x^i, \theta^i, \tilde{\varphi}^i, \tilde{\Delta}^i)$ is budgetary feasible, agent i will consume strictly more at the node ξ and will postpone the repayment of his debt until infinity by renewing it. Moreover, since (φ^i, Δ^i) and $(\tilde{\varphi}^i, \tilde{\Delta}^i)$ generate the same default, the disutility that agent i suffers is the same for these two couples. That is, if there exists $\hat{\varphi}^i$ for which equations (5) and (6) are possible, an agent with monotone preferences can always improve upon any budget feasible plan. Then, choosing such a portfolio $\hat{\varphi}^i$ the agent i will resort to Ponzi schemes and therefore his maximization problem has no solution.

Remark 3.1 In view of (5) and (6), Ponzi schemes are possible if there exists a node $\xi \in D$ such that form node ξ onwards, there is always some asset whose price exceeds the respective collateral costs, that is:

$$\forall \sigma \in D(\xi),\ \exists j \in J(\sigma):\ p(\sigma)C^j(\sigma) - q_j(\sigma) < 0. \tag{7}$$

In Araujo et al. (2002), since the utility penalties were absent, the returns from the joint operation of borrowing and securing the short–sale were always non–negative. By non–arbitrage, it followed that the borrowed value had to be less than or equal to the collateral cost, implying that Condition (7) would never hold. But, in our model, the borrowed values can be greater than the value of the constituted collateral, and in order to obtain an equilibrium, we need to introduce some suitable assumptions which avoid that (7) occurs.

4 The Assumptions and the Main Result.

We make on $\mathcal{E}$ the following assumptions:

Assumption [U]. $\forall i \in I,\ \forall \xi \in D$, the function $v^i_\xi : \mathbb{R}^G_+ \longrightarrow \mathbb{R}$ is continuous, monotone[1] and concave with $v^i_\xi(0) = 0$. Moreover, $\forall i \in I,\ \forall \alpha \in \mathbb{R}^G_+,\ \sum_{\xi \in D} v^i_\xi(\alpha)$ is finite.

Assumption [W]. $\exists W \in \mathbb{R}_{++} : \forall i \in I,\ \forall \xi \in D,\ \sum_{g \in G} \omega^i(\xi, g) \leq W.$

Assumption [D]. The depreciation structure of the commodities is given by: $[Y(\xi)] = [\text{diag}[a(\xi,g)]]_{g \in G}$ and there exists $k \in (0,1)$ such that for each node $\xi \in D,\ \max_{g \in G}\{a(\xi,g)\} \leq k$.

Assumptions [U] and [W] are classical in such models (cf. Araujo et al. (2002) for instance). Assumption [D] was also made by Araujo et al. (2002). In fact, this assumption is needed to guarantee that at each node, the aggregate initial endowment accumulated until this node is uniformly bounded from above along the event-tree.

Now, we introduce the following assumption on the utility penalties:

Assumption [P]. For each node $\xi \in D$, for each agent $i \in I$, there exists a date $t_{(i)} > t(\xi)$ such that $\forall \sigma :\ t(\sigma) = t_{(i)}$, if $\varphi^i_j(\sigma^-) \leq \frac{1}{\|C^j(\sigma^-)\|_1} \frac{W\mathcal{I}}{1-k}$, then for any $p(\sigma) > 0$, one has:

$$\sum_{j \in J(\sigma^-)} \lambda^i_j(\sigma) \frac{[p(\sigma)A^j(\sigma)\varphi^i_j(\sigma^-) - D^i_j(\sigma)]}{p(\sigma)b(\sigma)} \leq v^i_\sigma(\omega^i(\sigma)), \tag{8}$$

where $D^i_j(\sigma) := \min\{p(\sigma)A^j(\sigma)\varphi^i_j(\sigma^-), p(\sigma)Y(\sigma)C^j(\sigma^-)\varphi^i_j(\sigma^-)\}$.

Assumption [P] requires that at each date, there is some date in the future in which the penalties will not be too harsh. More precisely, it requires that at each date, there is some date in the future in which for each agent the default penalties for a maximal

[1]For each $x,\ y$ in $\mathbb{R}^G_+,\ y > x \implies v^i_\xi(y) > v^i_\xi(x)$.

default (i.e. when he pays only the minimum between the value of his debt and the value of the depreciated collateral, at an attainable short–sale) is less than the utility from the consumption of this node's endowment.

Remark 4.1 Note that if we can find a lower bound, $\underline{p}(\sigma)$, for the sum of commodity prices $\sum_{g\in G} p(\sigma,g)$, then [P] holds if the following condition, on the penalty rates, is satisfied:

$$\sum_{j\in J(\sigma^-)} \lambda_j^i(\sigma) \leq \underline{p}(\sigma)\,\underline{b}(\sigma)\,\frac{(1-k)\,\|C^j(\sigma^-)\|_1}{W\mathcal{I}\max_{g\in G} A_g^j(\sigma)}\,v_\sigma^i(\omega^i(\sigma)), \text{ where } \underline{b}(\sigma)=\min_{g\in G} b(\xi,g).$$

Such a lower bound, $\underline{p}(\sigma)$, for the sum of commodity prices, at each node, exists for all price plans compatible with Euler equations (that is, compatible with optimizing on current node variables, given values at the preceding node and at the immediate successors), see Proposition 5.2 (iii) below.

Moreover, we make on $\mathcal{E}$ the following survival assumption:

Assumption [S].

(i) For the initial node ξ_0, $\omega^i(\xi_0) \gg 0$, $\forall i \in I$.

(ii) For any node $\xi > \xi_0$, $Y(\xi) \neq 0$ and if $a(\xi,g)=0$ then $\omega^i(\xi,g)>0$.

That is, at any node at least one commodity does not completely depreciate and, for any commodity g, we require endowments to be positive at the initial node and also at any node where commodity g completely depreciates. In particular, commodities that are durable throughout the entire infinite tree are only required to have interior individual endowments at the origin of the tree. A perishable good must have positive individual at nodes where it perishes. This assumption guarantees that for each agent and at each node the initial endowment accumulated until this node is positive.

Theorem 4.1 *Under the assumptions [U], [W] , [D], [P] and [S], the economy $\mathcal{E}$ has a non–trivial equilibrium.*

5 Equilibria in truncated economies.

Let $\mathcal{E}^T$ be the truncated economy associated to the original economy $\mathcal{E}$, which has the same characteristics than $\mathcal{E}$, but where we suppose that agents are constrained to stop their exchange of goods at period T and their trade of assets at period $T-1$. Formally,

for each $T>0$, let us define the following sets:

$$\Pi^{T-1} := \left\{ (p,q) \in \mathbb{R}_+^{D^T\times G} \times \prod_{\xi\in D^T} \mathbb{R}^{\iota(\xi)} \;\middle|\; \begin{array}{l} \forall\xi:\ t(\xi)<T,\ \|p(\xi)\|_1+\|q(\xi)\|_1=1, \\ \forall\xi:\ t(\xi)=T,\ \|p(\xi)\|_1=1. \end{array} \right\},$$

$$\mathcal{K}^T := [0,1]^{\left(\sum_{\xi\in D^T}\iota(\xi)\right)},$$

and for each $i\in I$,

$$X^{iT} = \{(x^i(\xi),\ \xi\in D)\in X^i \mid \forall\xi : t(\xi)>T,\ x^i(\xi)=0\},$$

$$Z^{iT} = \{(z^i(\xi),\ \xi\in D)\in X^i \mid \forall\xi : t(\xi)\geq T,\ \theta^i(\xi)=\varphi^i(\xi)=0\}.$$

Moreover, given $(p,q,K)\in\Pi^{T-1}\times\mathcal{K}^T$, the budget set, $B^{iT}(p,q,K)$, of an agent $i\in I$ for the truncated economy is defined by the set of (x,z,Δ) such that $x^i\in X^{iT}$, $z^i\in Z^{iT}$, (2) holds at $\xi=0$ and (3)-(4) hold at all the other nodes.

Moreover, for each agent $i\in I$, the utility function U^{iT} for each truncated economy $\mathcal{E}^T$ is defined as follows for a price process such that $p(\xi)>0$ at any node ξ:

$$U^{iT}(x^i,\theta^i,\varphi^i,\Delta^i) := \sum_{\xi\in D^T} v_\xi^i(\tilde{x}^i(\xi)) - \sum_{\xi\in D^T\setminus\{0\}}\sum_{j\in J(\xi^-)} \lambda_j^i(\xi)\frac{[p(\xi)A^j(\xi)\varphi_j^i(\xi^-)-\Delta_j^i(\xi)]^+}{p(\xi)b(\xi)}. \tag{9}$$

Definition 5.1 *[Equilibria of the truncated economies]*
An equilibrium of $\mathcal{E}^T$ is a collection $(\overline{p}^T,\overline{q}^T,\overline{K}^T,(\overline{x}^{iT},\overline{\theta}^{iT},\overline{\varphi}^{iT},\overline{\Delta}^{iT})_{i\in I})$ such that $\overline{p}^T(\xi)>0$ at any node $\xi\in D^T$ and verifying:

(a) For each agent $i\in I$, $(\overline{x}^{iT},\overline{\theta}^{iT},\overline{\varphi}^{iT},\overline{\Delta}^{iT})\in \text{Argmax } U^{iT}(x,\theta,\varphi,\Delta)$ over $B^{iT}(\overline{p}^T,\ \overline{q}^T,\ \overline{K}^T)$,

(b) Conditions (ii)–(v) of Definition 2.2 hold at $(\overline{x}^T,\overline{\theta}^T,\overline{\varphi}^T,\overline{\Delta}^T)$ for $\xi\in D^T$, with $\overline{\varphi}^T(\xi)=0$ when $t(\xi)=T$.

An equilibrium of $\mathcal{E}^T$ is said to be non–trivial if, in addition, we have:

(c) For any (ξ,j), one has $(\theta_j(\xi),\varphi_j(\xi))$ is different from zero or $K^j(\xi)>0$.

For $n>0$, let us define the function U_n^i as follows:

$$U_n^{iT}(x^i,\theta^i,\varphi^i,\Delta^i) := \sum_{\xi\in D^T} v_\xi^i(\tilde{x}^i(\xi)) - \sum_{\xi\in D^T\setminus\{0\}}\sum_{j\in J(\xi^-)} \tilde{\lambda}_j^{i,n}(\xi)[p(\xi)A^j(\xi)\varphi_j^i(\xi^-)-\Delta_j^i(\xi)]^+, \tag{10}$$

where $\tilde{\lambda}_j^{i,n}(\xi) = \lambda_j^i(\xi)\ \frac{1}{\max\{1/n,\, p(\xi)\, b(\xi)\}}$.

Note that, by definition, in equilibrium, $p(\xi) > 0$ for each node $\xi \in D^T$, and since $b(\xi) \gg 0$, one gets $\tilde{\lambda}_j^{i,n}(\xi) = \lambda_j^i(\xi)\ \frac{1}{p(\xi)\, b(\xi)}$, for n large enough.
The modified utility functions U_n^{iT} will be taken as describing preferences in an auxiliary truncated economy. In the appendix, we start by addressing existence of equilibrium for this auxiliary truncated economy and showing that in this equilibrium the sum of commodity prices have positive lower bounds independent of n, node by node, implying that for n large enough, this is also an equilibrium for the original truncated economy. The following intermediary results are stated for the modified function U_n^{iT}.

Proposition 5.1 *Let* $(p, q, K) \in \prod_{\xi \in D^T} \left(\Delta^{G+\iota(\xi)-1} \times [0,1]^{\iota(\xi)}\right)$ *and let* $(x, \theta, \varphi, \Delta)$ *in* $\operatorname{Argmax}\{U_n^{iT}(\mathcal{Z}) \ :\ \mathcal{Z} \in B^i(p, q, K)\}$. *Then, the Kuhn–Tucker necessary conditions hold at each node* ξ, *that is, there exist multipliers* $\mu(\xi)$ *and* $\rho_j(\xi)$ $(j \in J(\xi))$, *for constraints (3) and (4), respectively, together with vectors of supergradients* $v'_\xi(x(\xi) + C(\xi)\varphi(\xi)) \in \partial v_\xi(x(\xi) + C(\xi)\varphi(\xi))$ *and* $(\lambda_j^i(\xi)\, d_j(\xi))$ *(where* $d_j(\xi) \in [0,1]$*) supergradients of the penalty with respect to default, such that for* $\tilde{x}^i = x^i + C\,\varphi^i$, *we have:*

$$\forall g \in G,\ v'_\xi(\tilde{x}^i(\xi), g) - \mu(\xi) p(\xi, g) + \sum_{\eta \in \xi^+} \mu(\eta) p(\eta, g) Y(\eta, g) \leq 0, \tag{11}$$

$$\forall g \in G,\ \left[v'_\xi(\tilde{x}^i(\xi), g) - \mu(\xi) p(\xi, g) + \sum_{\eta \in \xi^+} \mu(\eta) p(\eta, g) Y(\eta, g)\right] \overline{x}(\xi) = 0, \tag{12}$$

$$\begin{aligned}\mu(\xi)\, p(\xi)\, C^j(\xi) \ \geq\ \sum_{\eta \in \xi^+} \Big[\ & \mu^i(\eta) p(\eta) Y(\eta) C^j(\xi) - \rho_j(\eta) \min\{p(\eta) A^j(\eta),\, p(\eta) Y(\eta) C^j(\xi)\} \\ & -\tilde{\lambda}_j^{i,n}(\eta)\ d_j(\eta)\ p(\eta) A^j(\eta)\ \Big] + v'_\xi\,(\tilde{x}(\xi))\, C^j(\xi) + \mu^i(\xi)\, q_j(\xi)\end{aligned} \tag{13}$$

$$\begin{aligned}\Big\{\mu(\xi)\, p(\xi)\, C^j(\xi) - \sum_{\eta \in \xi^+} \Big[\ & \mu^i(\eta) p(\eta) Y(\eta) C^j(\xi) - \rho_j(\eta) \min\{p(\eta) A^j(\eta),\, p(\eta) Y(\eta) C^j(\xi)\} \\ -\tilde{\lambda}_j^{i,n}(\eta)\ d_j(\eta)\ p(\eta) A^j(\eta)\ \Big] & - v'_\xi\,(\tilde{x}(\xi))\, C^j(\xi) - \mu^i(\xi)\, q_j(\xi)\Big\}\, \varphi_j^i(\xi) = 0.\end{aligned} \tag{14}$$

$$\left[-\mu(\xi) q_j(\xi) + \sum_{\eta \in \xi^+} \mu(\eta) K^j(\eta) p(\eta) A^j(\eta)\right] \leq 0. \tag{15}$$

$$\left[-\mu(\xi) q_j(\xi) + \sum_{\eta \in \xi^+} \mu(\eta) K^j(\eta) p(\eta) A^j(\eta)\right] \theta^j(\xi) = 0. \tag{16}$$

$$\tilde{\lambda}_j^{i,n}(\eta)\, d_j(\eta) - \mu_j(\eta) + \rho_j(\eta) \leq 0, \tag{17}$$

$$\left[\tilde{\lambda}_j^{i,n}(\eta)\, d_j(\eta) - \mu_j(\eta) + \rho_j(\eta)\right] \Delta_j^i(\eta) = 0, \tag{18}$$

with $d_j(\eta) = 1$ *if* $p(\eta) A^j(\eta) \varphi_j(\xi) > \Delta_j(\eta)$ *and* $d_j(\eta) \in [0,1]$ *otherwise.*

Proof. The previous result holds since Slater's condition is satisfied, as proven in the appendix, part II, Claim 7.1.

Let us denote $\overline{C}^j(\xi) = \max_{g \in G} C_g^j(\xi)$, $\overline{A}^j(\xi) = \max_{g \in G} A_g^j(\xi)$ and $\underline{b}(\xi) = \min_{g \in G} b(\xi, g) > 0$. Let $(v_{\xi,g}^i)'_+$ be the right derivative of v_ξ with respect to good g and let $M_{\xi,g} = (v_{\xi,g}^i)'_+(\frac{W\mathcal{I}}{1-k})$.

Proposition 5.2 *Let* $(p, q, K) \in \prod_{\xi \in D^T} \left(\Delta^{G+\iota(\xi)-1} \times [0,1]^{\iota(\xi)}\right)$ *such that* $p(\xi) \neq 0$ *for each* $\xi \in D$ *and let* $(x, \theta, \varphi, \Delta)$ *in* $\operatorname{Argmax}\{U_n^{iT}(\mathcal{Z}) \ :\ \mathcal{Z} \in B^i(p, q, K)\}$. *Under assumptions [U], [W], and [S], for each agent* $i \in I$, *there exist Lagrange multipliers* $(\mu^i(\xi))_{\xi \in D^T}$ *such that for each node* $\xi \in D^T$, *one has:*

(i) $\forall g \in G,\ \mu^i(\xi)\, p(\xi, g) \geq M_{\xi,g} > 0.$

(ii) $\frac{q_j(\xi)}{\sum_{g \in G} p(\xi,g)} \leq \overline{C}^j(\xi) + \frac{1}{\sum_{g \in G} M_{\xi,g}} \sum_{\eta \in \xi^+} \frac{\lambda_j(\eta)\, \overline{A}^j(\xi')}{\underline{b}(\xi)} \equiv \widehat{q}_j(\xi).$

(iii) $\sum_{g \in G} p(\xi, g) \geq \frac{1}{1 + \sum_{j \in J(\xi)} \widehat{q}_j(\xi)} \equiv \underline{p}(\xi) > 0.$

(iv) $\mu^i(\xi)\left[p(\xi)\, x(\xi) + \left(p(\xi)\, C^j(\xi) - q_j(\xi)\right) \varphi(\xi) + q_j(\xi)\, \theta(\xi)\right] \ \leq\ \sum_{\eta \geq \xi} v_\eta^i\left(x(\eta) + C^j(\eta) \varphi(\eta)\right)$

Corollary 5.1 *Under the assumptions of Proposition 5.2,*

(1) if $\omega^i(\xi) \gg 0$ *and* $\Delta^i(\xi) = \min\{p(\xi) A(\xi),\ p(\xi) Y(\xi) C(\xi^-)\}\, \varphi^i(\xi^-)$, *then*

$$\mu^i(\xi) \leq \frac{1 + \sum_{j \in J(\xi)} \widehat{q}_j(\xi)}{\min_{(i,g) \in I \times G} \omega^i(\xi, g)} \sum_{\eta \geq \xi} v_\eta^i\,(\frac{W\mathcal{I}}{1-k}) \equiv \overline{\mu}^i(\xi). \tag{19}$$

(2) In general, one has:

$$\mu^i(\xi)\left[p(\xi)\, \omega^i(\xi) + p(\xi) Y(\xi) x^i(\xi^-) + \sum_{j \in J_1(\xi^-)} \left(p(\xi)\, Y(\xi)\, C^j(\xi^-) \varphi_j^i(\xi^-) - \Delta_j^i(\xi)\right)\right] \leq$$

$$\left[\sum_{\eta \geq \xi} v_\eta^i\,(\tfrac{W\mathcal{I}}{1-k}) + \sum_{j \in J(\xi)} \frac{\lambda_j^i(\xi) \overline{A}^j(\xi)\, W\,\mathcal{I}}{\overline{C}^j(\xi)(1-k)\, \underline{b}(\xi)}\right] \equiv \alpha^i(\xi),$$

(3) For any asset $j \in J(\xi)$ and for any $g \notin S(\xi, j)$ (where $S(\xi, j)$ was defined in Remark 2.1), one has:

$$\frac{p(\xi,g)}{\sum_{\kappa\in S(\xi,j)} p(\xi,\kappa)} \text{ has an upper bound, which will be denoted by } \widehat{p}^T(\xi,g) \quad (20)$$

and,

$$\sum_{\kappa\in S(\xi,j)} p(\xi,\kappa) \geq \Big(1 + \sum_{j\in J(\xi)} \widehat{q}_j(\xi) + \sum_{j\in J(\xi^-)} \sum_{\kappa\in S(\xi,j)} \widehat{p}^T(\xi,\kappa)\Big)^{-1} \equiv \delta_j(\xi). \quad (21)$$

Note that the bounds obtained in Proposition 5.2, (i), (ii) and (iii), and in Corollary 5.1, (1) and (3), do not depend on the particular optimal solution or on the time truncation T. Thus, these bounds must be satisfied by any price process compatible with individual optimality in problems with any truncated finite horizon. These bounds do not depend either on the parameter n of the modified utility function U_n^{iT} and, for this reason, since item (iii) of Proposition 5.2 holds in equilibrium of auxiliary economies (with modified U_n^{iT}), an equilibrium of the auxiliary economy, for n large enough, will be shown to be also an equilibrium for the original truncated economy. Notice also that it is item (3) in Corollary 5.1 that will play a crucial role in guaranteeing the non–triviality of equilibria.

Proposition 5.3 *Under assumptions [U] and [S], each truncated economy $\mathcal{E}^T$ has a non–trivial equilibrium $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})_{i\in I})$.*

Proof. See appendix.

Corollary 5.2 *Let $(p, q, K) \in \prod_{\xi\in D} \left(\Delta^{G+\iota(\xi)-1} \times [0,1]^{\iota(\xi)}\right)$ such that $p(\xi) > 0$ for each $\xi \in D$ and let $(x, \theta, \varphi, \Delta)$ in $\operatorname{Argmax}\{U^i(\mathcal{Z}) : \mathcal{Z} \in B^i(p, q, K)\}$. If Assumption [S] holds and the short-sales plan is feasible with respect to the aggregate resources available for collateralization (i.e.: $\varphi_j^i(\xi) \leq \frac{WI}{(1-k)\overline{C}^j(\xi^-)}$), then conditions (11)–(18) of Proposition 5.1, items (i)–(iv) of Proposition 5.2 and items (1)–(3) of Corollary 5.1 hold for the optimum $(p, q, K, x, \theta, \varphi, \Delta)$ of the infinite–horizon problem.*

Proof. Take a sequence of maximizers of the optimization problems with horizon T_k at prices (p, q, K) fixed. This sequence together with the respective sequence of multipliers μ_k, ρ_k and coefficients d_k has a subsequence which converges node by node (see proof

of Theorem 4.1, in section 6 below, and Corollary 5.1 (2)). At the cluster point, the conditions of Proposition 5.1 hold (the argument done by Araujo et al (2005) in the context without utility penalties applies here too, using the bounds for $\mu(\xi)$ given by Corollary 5.1 (2)).

6 Existence of equilibrium for the original economy.

In this section, we prove first that, as time horizon increases, the sequence of the equilibria for the truncated economies has a cluster point and, secondly, that individual optimality holds at this cluster point.

Under assumptions [U], [W] and [D], one has for each node $\xi \in D^T$:

$$\sum_{(i,g)\in I\times G} [\overline{x}^{iT}(\xi,g) + \sum_{j\in J(\xi)} C_g^j(\xi)\,\overline{\varphi}_j^{iT}(\xi)] \leq W\mathcal{I}\sum_{n=0}^{+\infty} k^n = \frac{W\mathcal{I}}{1-k} < +\infty, \quad (22)$$

$$\sum_{i\in I} \overline{\varphi}_j^{iT}(\xi) \leq \frac{1}{c^j(\xi)}\,\frac{W\mathcal{I}}{1-k}, \quad (23)$$

$$\sum_{i\in I} \overline{\theta}_j^{iT}(\xi) \leq \frac{1}{c^j(\xi)}\,\frac{W\mathcal{I}}{1-k}, \quad (24)$$

$$\sum_{i\in I} \overline{\Delta}_j^i(\xi) \leq \|A^j(\xi)\|_1 \frac{1}{c^j(\xi^-)}\,\frac{W\mathcal{I}}{1-k}, \quad (25)$$

where $c^j(\xi)$ was already defined in Remark 2.1.

Note that the bounds defined found in (22), (23), (24) and (25) depend on the node but not on the horizon of the truncated economy (so that as T goes to infinity, the sequence of truncated equilibrium variables associated with each node is a bounded sequence).

Then, in view of conditions (22)–(25) and the countability of the set of nodes D, we get, via a diagonalization procedure as in Araujo and al. (2002), a sequence $\{T_k\}_{k\in\mathbb{N}}$ such that $((\overline{x}^{T_k}, \overline{\theta}^{T_k}, \overline{\varphi}^{T_k}, \overline{\Delta}^{T_k}), \overline{p}^{T_k}, \overline{q}^{T_k}, \overline{K}^{T_k})$ which converges, at each node, to some $((\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\Delta}), \overline{p}, \overline{q}, \overline{K})$.

Optimality of the cluster point.

Lemma 6.1 $\sum_{g\in G} \overline{p}(\xi, g)$ *is bounded from below by $\underline{p}(\xi)$, given by inequality (iii) in Proposition 5.2, and therefore, U^i is well defined at $\overline{p}$, for each $i \in I$.*

Proof. The previous lemma follows by convergence, node by node, of p to $\overline{p}$ and since the lower bound $\underline{p}(\xi)$ does not depend neither on the time horizon T nor the particular optimal solution for the truncated optimal problem, and does not even depend on the parameter n.

Proposition 6.1 *For each agent $i \in I$, $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$ is optimal in $B^i(\overline{p}, \overline{q}, \overline{K})$.*

Proof of Proposition 6.1.
By contraposition, let us assume that $\exists i \in I$ and $(\widehat{x}^i, \widehat{\theta}^i, \widehat{\varphi}^i, \widehat{\Delta}^i) \in B^i(\overline{p}, \overline{q}, \overline{K})$ such that:

$$U^i(\widehat{x}^i, \widehat{\theta}^i, \widehat{\varphi}^i, \widehat{\Delta}^i) - U^i(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) > 0.$$

Then, $\exists \overline{T} : \ \forall T > \overline{T}$,
$$\sum_{\xi \in D^T} v^i_\xi(\widehat{x}^i(\xi)) - \sum_{\xi \in D^T} \sum_{j \in J(\xi^-)} \lambda^i_j(\xi) \frac{[\overline{p}(\xi) A^j(\xi) \varphi^i_j(\xi^-) - \widehat{\Delta}^i_j(\xi)]}{\overline{p}(\xi)\, b(\xi)} > U^i(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i).$$

Let us fix $\tilde{T} > \overline{T}$ such that for each node $\sigma : \ t(\sigma) = \tilde{T} + 1$ the default penalties satisfy Equation (8) of Assumption [P]. For each $y := (y(\xi), \ \xi \in D)$, let us define the following correspondence:

$$\psi^{\tilde{T}}(y) := \{(x(\xi), \theta(\xi), \varphi(\xi), \Delta(\xi), \ \xi \in D^{\tilde{T}}) \mid U^{i\tilde{T}}(x, \theta, \varphi, \Delta) > U^i(y)\}.$$

Now, $\psi^{\tilde{T}}$ is lower hemicontinuous with respect to the product topology on $L^\infty(D)$. In fact, U^i is Mackey upper semicontinous and concave, hence weak star upper semicontinuous. Moreover, $\psi^{\tilde{T}}(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\Delta})$ is open and convex.
On the other hand, for each (p, q, K), let us define:

$$\beta^{\tilde{T}}(p, q, K) = \{((x(\xi), \theta(\xi), \varphi(\xi), \Delta(\xi), \ \xi \in D^{\tilde{T}})) \text{ satisfying the budget constraints (3) and (4) of the original economy } \mathcal{E} \text{ at } (p, q, K)\}.$$

Since $(\widehat{x}, \widehat{\theta}, \widehat{\varphi}, \widehat{\Delta}) \in \beta^{\tilde{T}}(\overline{p}, \overline{q}, \overline{K}) \cap \psi^{\tilde{T}}(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\Delta})$ and $\beta^{\tilde{T}}(\overline{p}, \overline{q}, \overline{K}) \cap \psi^{\tilde{T}}(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\Delta})$ is lower semicontinuous (e.g., Hildenbrand (1974), p.35, Prob.6 (1)), for the topology of convergence node by node on the countable set D, one gets the existence of $T^\star$ and a sequence $(\widehat{x}^T, \widehat{\theta}^T, \widehat{\varphi}^T, \widehat{\Delta}^T)$ converging to $(\widehat{x}, \widehat{\theta}, \widehat{\varphi}, \widehat{\Delta})$ such that $\forall T \geq T^\star$,
$(\widehat{x}^T, \widehat{\theta}^T, \widehat{\varphi}^T, \widehat{\Delta}^T) \in \beta^{\tilde{T}}(\overline{p}^T, \overline{q}^T, \overline{K}^T) \cap \psi^{\tilde{T}}(\overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T)$.

Let us distinguish the following cases:

(i) If $T^\star \leq \tilde{T}$. Taking $T = \tilde{T} + 1$, we get:

$$U^{i\,\tilde{T}}(\hat{x}^{\tilde{T}+1}, \hat{\theta}^{\tilde{T}+1}, \hat{\varphi}^{\tilde{T}+1}, \hat{\Delta}^{\tilde{T}+1}) > U^{i\,(\tilde{T}+1)}(\overline{x}^{\tilde{T}+1}, \overline{\theta}^{\tilde{T}+1}, \overline{\varphi}^{\tilde{T}+1}, \overline{\Delta}^{\tilde{T}+1})$$

and $(\hat{x}^{\tilde{T}+1}, \hat{\theta}^{\tilde{T}+1}, \hat{\varphi}^{\tilde{T}+1}, \hat{\Delta}^{\tilde{T}+1})$ satisfies the budget constraints of the truncated economy $\mathcal{E}^{\tilde{T}}$ at $(\overline{p}^{\tilde{T}+1}, \overline{q}^{\tilde{T}+1}, \overline{K}^{\tilde{T}+1})$.

(ii) If $T^\star > \tilde{T}$. Taking $T = T^\star$, we get

$$U^{i\,\tilde{T}}(\hat{x}^{T^\star}, \hat{\theta}^{T^\star}, \hat{\varphi}^{T^\star}, \hat{\Delta}^{T^\star}) > U^{i\,T^\star}(\overline{x}^{T^\star}, \overline{\theta}^{T^\star}, \overline{\varphi}^{T^\star}, \overline{\Delta}^{T^\star})$$

and $(\hat{x}^{T^\star}, \hat{\theta}^{T^\star}, \hat{\varphi}^{T^\star}, \hat{\Delta}^{T^\star})$ satisfies the budget constraints of the truncated economy $\mathcal{E}^{\tilde{T}}$ at $(\overline{p}^{T^\star}, \overline{q}^{T^\star}, \overline{K}^{T^\star})$.

Let us construct a plan $(x^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT})$, for consumer i, satisfying the budget constraints of the economy $\mathcal{E}^T$ at $(\overline{p}^T, \overline{q}^T, \overline{K}^T)$ and such that $U^{i\,T}(x^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT}) = U^{\tilde{T}+1}(x^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT}) > U^{i\,T}(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$, contradicting the fact that $(\overline{p}^T, \overline{q}^T, \overline{K}^T, \overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T)$ is an equilibrium for $\mathcal{E}^T$ (in case (i), we take $T = \tilde{T} + 1$ and in case (ii) we take $T = T^\star$).

$$x^{iT}(\xi) = \begin{cases} \widehat{x}^{iT}(\xi) & \text{if } t(\xi) \leq \tilde{T} \\ \omega^i(\xi) & \text{if } t(\xi) = \tilde{T} + 1 \\ 0 & \text{if } t(\xi) > \tilde{T} + 1 \end{cases}, \qquad \theta^{iT}(\xi) = \begin{cases} \widehat{\theta}^{iT}(\xi) & \text{if } t(\xi) \leq \tilde{T} \\ 0 & \text{if } t(\xi) \geq \tilde{T} + 1 \end{cases}$$

$$\varphi^{iT}(\xi) = \begin{cases} \widehat{\varphi}^{iT}(\xi) & \text{if } t(\xi) \leq \tilde{T} \\ 0 & \text{if } t(\xi) \geq \tilde{T} + 1 \end{cases}, \quad \Delta^{iT}_j(\xi) = \begin{cases} \widehat{\Delta}^{iT}_j(\xi) & \text{if } t(\xi) \leq \tilde{T}, \\ D^{iT}_j(\xi), & \text{if } t(\xi) = \tilde{T} + 1, \\ 0 & \text{if } t(\xi) > \tilde{T} + 1. \end{cases}$$

where $D^{iT}_j(\xi) := \min\{\overline{p}^T(\xi)\, A^j(\xi)\, \widehat{\varphi}^{iT}_j(\xi^-), \overline{p}^T(\xi)\, Y(\xi)\, C^j(\xi^-)\, \widehat{\varphi}^{iT}_j(\xi^-)\}$.

Clearly, $(x^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT})$ satisfies the budget constraints up to time $\tilde{T}$ of the truncated economy $\mathcal{E}^T$. Let us see that the remaining budget constraints are satisfied.
At each node ξ such that $t(\xi) = \tilde{T} + 1$, one gets:
$$\overline{p}^T(\xi) \cdot (x^{iT}(\xi) - \omega^i(\xi)) + \sum_{j \in J(\xi^-)} \Delta^{iT}_j(\xi) - \overline{p}^T(\xi)[Y(\xi) x^{iT}(\xi^-) + Y(\xi) C(\xi^-) \varphi^{iT}(\xi^-)$$
$$+ \sum_{j \in J(\xi^-)} \overline{K}^T_j(\xi) A^j(\xi) \theta^{iT}_j(\xi^-)] = \sum_{j \in J(\xi^-)} \min\{\overline{p}^T(\xi) A^j(\xi), \overline{p}^T(\xi) Y(\xi) C^j(\xi^-)\}\, \widehat{\varphi}^{iT}_j(\xi^-)$$
$$- \overline{p}^T(\xi)[Y(\xi) \widehat{x}^{iT}(\xi^-) + Y(\xi) C(\xi^-) \widehat{\varphi}^{iT}(\xi^-) + \sum_{j \in J(\xi^-)} \overline{K}^T_j(\xi) A^j(\xi) \widehat{\theta}^{iT}_j(\xi^-)] \leq 0.$$
For each node ξ such that $t(\xi) > \tilde{T} + 1$, it is obvious that $(x^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT})$ satisfies the budget constraints.
With these changes, agent i will pay, at the node $\sigma : t(\sigma) = \tilde{T} + 1$, $\Delta^{iT}_j(\sigma)$ and paid $\widehat{\Delta}^i_j(\sigma)$ before, with $\Delta^{iT}_j(\sigma) = \min\{\overline{p}^T(\sigma) A^j(\sigma) \varphi^{iT}_j(\sigma^-), \overline{p}^T(\sigma) Y(\sigma) C^j(\sigma^-) \varphi^{iT}_j(\sigma^-)\}$
$\leq \widehat{\Delta}^i_j(\sigma)$. Then, he will suffer a higher utility penalty because his default increased, and therefore he will loose some utility. The degree of severity of the default penalties will determine if this agent prefers these changes. As we will see later on, Assumption [P], that assumes that the default penalties are not too harsh, guarantees that the agent

still prefers these modified allocations (although he would be more penalized). Indeed, for each $T \geq T^\star$, one has:

$$\sum_{t(\xi)=\tilde{T}+1}^{T} v^i_\xi(x^{iT}(\xi)) - \sum_{t(\xi)=\tilde{T}}^{T}\sum_{j\in J(\xi^-)} \lambda^i_j(\xi)\frac{[\overline{p}^T(\xi)A^j(\xi)\varphi^{iT}_j(\xi^-)-\Delta^{iT}_j(\xi)]}{\overline{p}^T(\xi)\,b(\xi)} =$$

$$\sum_{\{\xi:t(\xi)=\tilde{T}+1\}}\Big(v^i_\xi(\omega^i(\xi)) - \sum_{j\in J(\xi^-)}\lambda^i_j(\xi)\frac{\left[\overline{p}^T(\xi)A^j(\xi)\widehat{\varphi}^{iT}_j(\xi^-)-\Delta^{iT}_j(\xi)\right]}{\overline{p}^T(\xi)\,b(\xi)}\Big) =$$

$$\sum_{\{\xi:t(\xi)=\tilde{T}+1\}}\Big(v^i_\xi(\omega^i(\xi)) - \sum_{j\in J(\xi^-)}\lambda^i_j(\xi)\frac{\left[\overline{p}^T(\xi)A^j(\xi)\widehat{\varphi}^{iT}_j(\xi^-)-D^{iT}_j(\xi)\right]}{\overline{p}^T(\xi)\,b(\xi)}\Big).$$

Let $\beta \in]0,1[$ and let us define:

$$(x'^i, \theta'^i, \varphi'^i, \Delta'^i) := \beta\,(x^T, \theta^T, \varphi^T, \Delta^T) + (1-\beta)\,(\overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T).$$

In view of Equation (23), and since the two vectors of the right hand side of the above equality have zero coordinates beyond date T, one can choose β, close to zero, such that: $\varphi'^i(\xi) \leq \frac{1}{\|C^j(\xi)\|_1}\frac{W\mathcal{I}}{1-k}$ and then using Assumption [P] and Assumption [U], one gets:

$$U^{iT}(x'^i, \theta'^i, \varphi'^i, \Delta'^i) > U^{iT}(\overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T)$$

and

$$(x'^i, \theta'^i, \varphi'^i, \Delta'^i) \in B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T),$$

which contradicts the optimality of $(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$ in $B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T)$. □

7 Examples.

The purpose of this section is to show, through two examples, that if Assumption [P] is not satisfied, the agents end up doing Ponzi schemes. In these examples, we consider situations where the original promises are greater than the value of the depreciated collateral, at least far along the event tree.

Example 1 This first example illustrates the occurrence of Ponzi schemes when default is not allowed (that is, when utility penalty is prohibitive). It illustrates in an extreme situation our claim that constituting collateral can not *per se* rule out Ponzi schemes: default must be possible. This first example supports, in a model where there are collateral requirements (although collateral is never seized as default is forbidden), the well known results (by Magill and Quinzii (1994), Hernandez and Santos (1996), Levine and Zame (1996)...) that equilibrium without default may not exist, if transversality or borrowing constraints are not imposed a priori.

Formally, Let us consider a model satisfying:

$$\forall \xi \in D,\ \forall j \in J(\xi),\ \forall \eta \in \xi^+,\ A^j(\eta) > Y(\eta)C^j(\xi), \tag{26}$$

and let $b(\xi) = 1,\ \forall \xi$. By Proposition 5.2, item (iv), assuming there is just one good, taken to be the numeraire, we have:

$$\mu^i(\xi_0) \leq \frac{1}{\omega^i(\xi_0)}\sum_{\xi\geq\xi_0} v^i_\xi(\frac{W\mathcal{I}}{1-k}) \equiv \overline{\mu}^i(\xi_0).$$

Now, equation (11) implies that $\mu^i(\xi)\,a(\xi) \leq \mu^i(\xi^-)$, for each node ξ.

Assume that $\lambda^i(\xi) > \Big(\prod_{\eta\in F_t}(a(\eta))^{-1}\Big)\overline{\mu}^i(\xi_0)$, where F_t is the history up to ξ.

Note that, in such a case, the condition (17) no longer holds if $d_j(\xi) = 1$ (that is, if default occurs). Thus, we must have $d_j(\xi) < 1$ and, therefore such penalties prohibit default.

Let (p, q, K) such that $q(\xi) \neq 0,\ \forall \xi \in D$. Let us consider an agent $i \in I$ and an allocation $(x^i, \varphi^i, \theta^i, \Delta^i) \in B^i(p, q, K)$. In fact, in view of the harshness of the utility penalties, the default will never occur in such a model, i.e.: $\forall \xi \in D,\ \forall j \in J(\xi),\ \forall \eta \in \xi^+$:

$$\Delta^i_j(\eta) = p(\eta)A^j(\eta)\varphi^i_j(\xi) \text{ and } K^j(\eta) = 1. \tag{27}$$

Then, in such a model, Equation (3) can be written as follows:

$$p(\xi)\cdot(x^i(\xi) - \omega^i(\xi)) + p(\xi)C(\xi)\varphi^i(\xi) + q(\xi)\cdot z^i(\xi)$$
$$\leq p(\xi)[Y(\xi)x^i(\xi^-) + Y(\xi)C(\xi^-)\varphi^i(\xi^-)] + \ p(\xi)A(\xi)z^i(\xi^-).$$

Now, let us fix a node $\xi \in D$ and let us consider the following changes on the portfolio of the agent i from node ξ onwards:

$$\widetilde{\varphi}^i(\sigma) = \varphi^i(\sigma) + \widehat{\varphi}^i(\sigma),\ \forall \sigma \in D(\xi).$$

It is easy to verify that, even in this case, the agent i can resort to Ponzi schemes if (5) and (6) holds. In this example since the utility penalties are prohibitive, (5) and (6) can occur simultaneously. Indeed, equation (26) implies that net returns $(pYC - pA)\varphi$ from constituting collateral and short selling must be negative, since default is ruled out. Hence, by absence of arbitrage, the net price of this joint operation will be negative also, and therefore a portfolio process can be found so that (5) and (6) hold and the agents end up doing Ponzi schemes.

In the following example, the penalty is not necessarily prohibitive as in the first example, but it violates Assumption [P] and agents end up also doing Ponzi schemes.

Example 2 Consider a deterministic economy with a single commodity (the numeraire) and a single asset whose collateral requirements C_t tend to zero as $t \longrightarrow +\infty$. Note that in this example too, we have also $b(\xi) = 1,\ \forall \xi$. Suppose asset real returns A_t are bounded from below by $\underline{A}$ and the depreciation factor Y_t is constant equal to k. Consider a consumer whose endowments are constant equal to W, the uniform upper bound on endowments of any consumer. The preferences of this consumer are given by:

$$U(x, \varphi, \Delta) = \sum_{t=0}^{\infty} \left[\beta^t (x_t + C_t \varphi_t) - \lambda_t (A_t \varphi_{t-1} - \Delta_t)^+\right],$$

where $\beta \in (0,1)$ and $\lambda_t = \beta^t \sigma$. Let us show that Assumption [P] is violated. Take $\varphi_t = \frac{W\mathcal{I}}{(1-k)\ C_t}$ and compute the penalty associated with the maximal default (that is, when the effective payment, Δ_t, is $\min\{A_t,\, k\, C_{t-1}\}\, \varphi_{t-1}$, which is equal to $k\, C_{t-1}\, \varphi_{t-1}$, for t large enough):

$$\begin{aligned}
\lambda_t (A_t - k\, C_{t-1})\, \varphi_{t-1} &= \beta^t \sigma (A_t - k\, C_{t-1}) \frac{W\mathcal{I}}{(1-k)\, C_{t-1}} \\
&= \beta^t\, W \frac{\sigma \mathcal{I}}{1-k} (\frac{A_t}{C_{t-1}} - k) \\
&\geq \beta^t\, W \frac{\sigma \mathcal{I}}{1-k} (\frac{\underline{A}}{C_{t-1}} - k).
\end{aligned}$$

Then, the penalty exceeds the utility $\beta^t W$ from the consumption of the current endowment, for t large enough.

Now, let us show that the consumer always pays more than the depreciated collateral if $\sigma = \frac{f\mathcal{I}}{(1-k)(1-\beta)}$, where $f > 1$. In fact, otherwise, by Corollary 5.1, the Lagrange multiplier of the date t budget constraint would satisfy:

$$\mu_t \leq \sum_{\tau \geq t} \frac{\beta^\tau\, (x_\tau + C_\tau \varphi_\tau)}{W} \leq \frac{\mathcal{I}\, \beta^t}{(1-k)(1-\beta)}.$$

Moreover, when $\Delta_t = k\, C_{t-1}\, \varphi_{t-1}$ default is positive, for t large enough, implying that the Kuhn–Tucker condition on Δ_t, given by $\lambda_t\, d_t + \rho_t \leq \mu_t$, becomes $\lambda_t \leq \mu_t$ (as $d_t = 1$), which is impossible, since $f > 1$. Then, $\Delta_t > \min\{A_t,\ k\, C_{t-1}\}\varphi_{t-1} = k\, C_{t-1} \varphi_{t-1}$, for t large enough. Hence, $\Delta_t > 0$ and therefore $\varphi_{t-1} > 0$ also, as $\Delta_t \leq A_t \varphi_{t-1}$. Notice also that $\rho_t = 0$ as $\Delta_t > \min\{A_t,\ kC_{t-1}\}\varphi_{t-1}$. Hence, the Kuhn–Tucker conditions on φ_t imply that:

$$\mu_t\, (C_t - q_t) = \mu_{t+1} k C_t - \lambda_{t+1} d_{t+1} A_{t+1} + \beta^t C_t,$$

where the last term on the right–hand side is the marginal utility from collateral consumption. Now, the Kuhn–Tucker conditions on x_t imply that $\mu_t \geq k\mu_{t+1} + \beta^t$. Since, $kC_t - A_{t+1} < 0$ and $\lambda_{t+1}\, d_{t+1} = \mu_{t+1}$ (as $(\lambda_{t+1} d_{t+1} - \mu_{t+1})\, \Delta_{t+1} = 0$ and $\Delta_{t+1} > 0$), we have

$$\begin{aligned}
\mu_t (C_t - q_t) &= \mu_{t+1}(kC_t - A_{t+1}) + \beta^t C_t \\
&\leq \beta^t (\beta\, k\, C_t - \beta A_{t+1} + C_t) \\
&\leq \beta^t [(\beta\, k + 1) C_t - \beta \underline{A}] < 0,
\end{aligned}$$

for t large enough. That is, the sufficient condition for occurrence of Ponzi schemes (see Remark 3.1) holds.

The penalties chosen in the current example are superior to μ_t for a maximal default (that is, when the effective payment is $\min\{A_t,\, k\, C_{t-1}\}\, \varphi_{t-1}$). However, these penalties may be inferior to μ_t if the default is not maximal. Therefore, in this example, the penalties lead to pay more than the minimum between the original promise and the depreciated collateral, but we claim that these penalties do not necessarily prohibit default. Indeed, it is sufficient to show that in general (and not just when $\Delta_t = \min\{A_t,\ kC_{t-1}\}\varphi_{t-1}$ as show in this example), it is possible to have $\lambda_t\, d_t \leq \mu_t$ with $d_t = 1$ (that is, default is allowed in this example). In fact, we have $\mu^i(\xi) \leq \frac{\mu^i(\xi^-)}{a(\xi)}$ and $\mu^i(\xi_0) \leq \frac{\sum_{\eta \geq \xi_0} v^i_\eta (\frac{W\mathcal{I}}{1-k})}{W} = \frac{\beta^0\ \mathcal{I}}{(1-k)(1-\beta)} = \frac{\mathcal{I}}{(1-k)(1-\beta)}$. Then, $\mu^i_t \leq \frac{\mathcal{I}}{(1-k)\,(1-\beta)k^t}$. But, $\lambda_t = \beta^t \sigma = \beta^t \frac{f\mathcal{I}}{(1-k)\,(1-\beta)}$, $f > 1$. Therefore, in this example, default is not ruled out if the following inequality holds:

$$\beta^t\, k^t \leq \frac{1}{f}. \tag{28}$$

Taking $f = 2$, $\beta = \frac{1}{2}$, $k = \frac{1}{2}$, the inequality (28) holds and, therefore, default may occur.

Remark 7.1 In Example 1, the penalties are harsh enough to prohibit default. In Example 2, the penalties induce that the default is not maximal (that is, the payment exceeds the minimum between the promise and the depreciated collateral) but, as shown above, the penalties chosen in Example 2 were not high enough to exclude default. However, these penalties were harsh enough to generate Ponzi schemes. That is, to allow for Ponzi schemes, the penalties do not need to be high enough so that default becomes prohibited, but it is just sufficient that the penalties are such that the payment exceeds the minimum between the promise and the depreciated collateral.

Concluding remarks.
Let us denote:

$$[\text{NA}] \quad \forall \xi \in D,\ \exists \sigma \in D^+(\xi) : \ \forall j \in J(\sigma) : \ p(\sigma)C^j(\sigma) - q_j(\sigma) \geq 0.$$

In our paper, we proved that under the assumptions [U], [W], [D] and [S], one has:

(i) Assumption [P] implies the existence of an equilibrium and we show also the converse for the economies in examples 1 and 2.

(ii) The existence of an equilibrium implies that [NA] is satisfied. Indeed, the existence of an equilibrium implies that Ponzi schemes are ruled out an therefore in view of (7), [NA] holds.

The items (i) and (ii) stated above can be summarized by the following diagram:

$$\begin{array}{ccc} [\text{NA}] & \Longleftarrow & \text{Ponzi schemes are ruled out} \\ & & \Uparrow \\ [\text{P}] & \Longrightarrow & \text{There exists an equilibrium} \end{array}$$

Let us notice that, in view of the previous diagram, Assumption [P] implies that the No–arbitrage condition [NA] is satisfied.

Appendix.

The Appendix has two parts. The first part is devoted to a proof of Proposition 5.2 and its corollary 5.1, while the second one is dedicated to prove Proposition 5.3.

Part I. Proof of Proposition 5.2.

Proof of assertion (i). Using (11), we have $\mu(\xi)p(\xi,g) \geq v'_{\xi,g}\Big(\overline{x}(\xi)+C(\xi)\overline{\varphi}(\xi)\Big) \geq M_{\xi,g}$, since $(v^i_{\xi,g})'_+$ is non–increasing. Now, $M_{\xi,g} > 0$, since $v^i_{\xi,g}$ is strictly increasing.

Proof of assertion (ii). Take inequality (13) and use inequality (17) to get

$$\mu(\xi)q_j(\xi) \leq \mu(\xi)p(\xi)C^j(\xi) + \sum_{\eta\in\xi^+} \widetilde{\lambda}^i_j(\eta)d(\eta)p(\eta)A^j(\eta). \tag{29}$$

Since $d_j(\xi) \leq 1$ and $\widetilde{\lambda}^i_j(\eta) = \frac{\lambda^i_j(\eta)}{\max\{\frac{1}{n}, p(\eta)\,b(\eta)\}}$, we get

$$q_j(\xi) \leq p(\xi)C^j(\xi) + \frac{1}{\mu(\xi)} \sum_{\eta\in\xi^+} \frac{\lambda^i_j(\eta)p(\eta)A^j(\eta)}{p(\eta)b(\eta)}. \tag{30}$$

Assertion (ii) then follows using assertion (i).

Proof of assertion (iii). This follows from (ii) and the fact that $(p(\xi), q(\xi))$ lies in the simplex $\Delta^{G+\iota(\xi)-1}$.

Proof of assertion (iv). Let us define $f(\xi) := \Big(x(\xi), \theta(\xi), \varphi(\xi), \Delta(\xi)\Big)$, $\tilde{x}(\xi) := x(\xi) + C(\xi)\varphi(\xi)$, and

$$(\text{def})^j_\xi\,(f(\xi)) = p(\xi)A^j(\xi)\varphi_j(\xi^-) - \Delta_j(\xi),$$
$$S_\xi(f(\xi)) := p(\xi)x(\xi) + (p(\xi)C(\xi) - q(\xi))\varphi(\xi) + q(\xi)\theta(\xi) + \Delta(\xi),$$
$$I_\xi(f(\xi)) := p(\xi)\omega(\xi) + p(\xi)Y(\xi)\tilde{x}(\xi^-) + \sum_{j\in J(\xi^-)} K_j(\xi)p(\xi)A^j(\xi)\theta(\xi^-),$$
$$\delta^j_\xi(f(\xi)) := \min\{p(\xi)A^j(\xi),\ p(\xi)Y(\xi)C^j(\xi^-)\}\ \varphi_j(\xi^-) - \Delta_j(\xi).$$

For each truncated economy, we define

$$L(f(\xi)_\xi) = \sum_{\xi\in D^T} v_\xi(\tilde{x}(\xi)) - \sum_{j\in J(\xi^-)} \widetilde{\lambda}^i_j(\xi)\left((\text{def})^j_\xi(f(\xi))\right)^+ - \mu_\xi\left[S_\xi(f(\xi)) - I_\xi(f(\xi))\right]$$
$$- \sum_{j\in J(\xi^-)} \rho_j(\xi)\,\delta^j_\xi(f(\xi)).$$

At equilibrium, $(\overline{f}^T(\xi)_\xi)$, of the truncated economy $\mathcal{E}^T$, one has:

$$L(\overline{f}^T(\xi)_\xi) - L(f(\xi)_\xi) \geq 0,\ \forall (f(\xi)_\xi) \geq 0. \tag{31}$$

Let us define $(\widehat{f}^T(\xi)_\xi)$ as follows:

$$\widehat{f}^T(\eta) = \begin{cases} \overline{f}^T(\xi) & \text{if } \eta \neq \xi \\ 0 & \text{otherwise} \end{cases}.$$

Then, using (31), one gets:

$$v_\xi\big(\overline{x}^T(\xi) + C(\xi)\,\overline{\varphi}^T(\xi)\big) + \sum_{j\in J(\xi^-)} [\widetilde{\lambda}_j(\xi)\,d_j(\xi) + \rho_j(\xi)]\,\overline{\Delta}^T(\xi) - \mu_\xi S_\xi(\overline{f}^T(\xi)) +$$
$$\sum_{\eta\in\xi^+} \left\{\mu(\eta)\Big(I_\eta(\overline{f}^T(\xi)) - p(\eta)\omega(\eta)\Big) - \sum_{j\in J(\xi)} \Big(\widetilde{\lambda}_j(\xi)\,d_j(\xi) + \rho_j(\eta)\Big)p(\eta)A^j(\eta)\varphi^i_j(\xi)\right\} \geq 0.$$

By (18) and since $p(\eta)A^j(\eta)\varphi_j(\xi) \geq \Delta_j(\eta)$, we get $\mu(\xi)\Big(S_\xi(\overline{f}^T(\xi)) - \sum_{j\in J(\xi)} \Delta_j(\xi)\Big) \leq v_\xi\big(\overline{x}^T(\xi) + C(\xi)\,\overline{\varphi}^T(\xi)\big) + \sum_{\eta\in\xi^+} \mu(\eta)\Big(S_\xi(\overline{f}^T(\xi)) - \sum_{j\in J(\xi)} \Delta_j(\eta)\Big)$. Then,

$$\mu(\xi)\Big(S_\xi(\overline{f}^T(\xi)) - \sum_{j\in J(\xi)} \Delta_j(\xi)\Big) \leq \sum_{\sigma\in D^T(\xi)} v_\sigma\big(\overline{x}^T(\sigma) + C(\sigma)\,\overline{\varphi}^T(\sigma)\big)$$

as claimed.

Proof of Corollary 5.1.
Proof of item (2). By item (iv) of Proposition 5.2, we have:

$$\mu^{iT}(\xi)\Big[S_\xi(f(\xi)) - \sum_{j\in J(\xi^-)} \Delta^i_j(\xi)\Big] \leq \sum_{\eta\geq\xi} v^i_\eta\Big(\overline{x}^T(\eta) + C^j(\eta)\overline{\varphi}^T(\eta)\Big).$$

Now, $\mu^{iT}(\xi)\, S_\xi(f(\xi)) = \mu^{iT}(\xi)\, I_\xi(f(\xi))$, and therefore one gets:

$$\mu^{iT}(\xi)\Big[p(\xi)\omega(\xi) + p(\xi)Y(\xi)x(\xi^-) + \sum_{j\in J_1(\xi^-)} p(\xi)Y(\xi)C^j(\xi^-)\varphi_j(\xi^-)\Big]$$

$\leq \sum_{\eta\geq\xi} v^i_\eta(\frac{W\mathcal{I}}{1-k}) + \sum_{j\notin J_1(\xi^-)} \Big(\widetilde{\lambda}_j(\xi)\; d_j(\xi) + \rho_j(\xi)\Big)\; \Delta_j(\xi)$, where:

$d_j(\xi) \leq 1$ and $\rho_j(\xi) = 0$, for $j \notin J_1(\xi^-)$. Now, $\Delta_j(\xi) \leq \overline{A}^j(\xi)\frac{W\mathcal{I}}{(1-k)\,\overline{C}^j(\xi^-)} \sum_{g\in G} p(\xi,g)$ and $\widetilde{\lambda}_j(\xi) \sum_{g\in G} p(\xi,g) \leq \frac{\lambda_j(\xi)}{b(\xi)}$.

Proof of item (3). Let $g \notin S(\xi,j)$ for some $j \in J(\xi^-)$.

(a) If g perishes at node ξ, that is $a(\xi,g) = 0$, we have by Assumption [S] that $\omega^i(\xi,g) > 0$. Then, by item(2), we have $\mu^i(\xi)p(\xi,g)\omega^i(\xi,g) \leq \alpha^i(\xi)$, and therefore, by Proposition 5.2, item (i), one gets:

$$\frac{p(\xi,g)}{\sum_{\kappa\in S(\xi,j)} p(\xi,g)} \leq \frac{\alpha^i(\xi)}{\omega(\xi,g) \sum_{\kappa\in S(\xi,j)} M_{\xi,\kappa}}. \tag{32}$$

(b) If g does not perish at node ξ, that is $a(\xi,g) > 0$, we have, by equation (11), that

$$\mu^i(\xi^-)p(\xi^-,g) \geq \mu^i(\xi)p(\xi,g)a(\xi,g).$$

Then, by Proposition 5.2, item (i), one gets:

$$\frac{p(\xi,g)}{\sum_{\kappa\in S(\xi,j)} p(\xi,\kappa)} \leq \frac{\mu^i(\xi^-)}{a(\xi,g) \sum_{\kappa\in S(\xi,j)} M_{\xi,\kappa}}.$$

Now, $\mu^i(\xi_0)$ has an upper bound $\overline{\mu}^i(\xi_0)$, as in item (1), since $\omega^i(\xi_0) \gg 0$ and at the initial node there are no financial payments to be made. At other nodes, by Assumption [S], item (iii), we know that $Y(\xi) \neq 0$ and, therefore,

$$\mu^i(\xi^-) \geq \mu^i(\xi^-) \sum_{g:\, a(\xi,g)\neq 0} p(\xi^-,g) \geq \mu^i(\xi) \sum_{g:\, a(\xi,g)\neq 0} p(\xi,g)\, a(\xi,g),$$

(again by equation (11)), which implies that:

$\mu^i(\xi) \leq \mu^i(\xi^-)\; (\gamma(\xi))^{-1}$, where $\gamma(\xi) = \Big(\min_{g:\, a(\xi,g)\neq 0} a(\xi,g)\Big)\Big(\sum_{g:\, a(\xi,g)\neq 0} p(\xi,g)\Big)$.

Let us show that the sum of the prices of durable goods, $\sum_{g:\, a(\xi,g)\neq 0} p(\xi,g)$, is bounded from below, node by node.
If ξ is such that there exists $\ell \in G$ such that $a(\xi,\ell) = 0$, then, by Assumption [S], $\omega^i(\xi,\ell) > 0$ and we get:

$$\frac{p(\xi,\ell)}{\sum_{\kappa:\, a(\xi,\kappa)\neq 0} p(\xi,g)} \leq \frac{\alpha^i(\xi)}{\omega(\xi,g) \sum_{\kappa:\, a(\xi,\kappa)\neq 0} M_{\xi,\kappa}} \equiv m_\ell\,(\xi).$$

Moreover, item (ii) of Proposition 5.2 can be adapted to yield:

$$\frac{\sum_{j\in J(\xi)} q_j(\xi)}{\sum_{\kappa:\, a(\xi,\kappa)\neq 0} p(\xi,\kappa)} \leq \sum_{j\in J(\xi)} \Big[\overline{C}^j(\xi) + \frac{1}{\sum_{\kappa:\, a(\xi,\kappa)\neq 0} M_{\xi,\kappa}} \sum_{\xi'\in\xi^+} \lambda^i_j(\xi')\overline{A}^j(\xi')\Big] \equiv m(\xi).$$

Then,

$$\sum_{\kappa:\, a(\xi,\kappa)\neq 0} p(\xi,\kappa) \geq \Big(1 + \sum_{\kappa:\, a(\xi,\kappa)=0} m_\kappa(\xi) + m(\xi)\Big)^{-1} \equiv D(\xi) > 0.$$

Let $\overline{\gamma}(\xi) = D(\xi) \min_{g:\, a(\xi,g)\neq 0} a(\xi,g)$, and we get $\mu^i(\xi) \leq \mu^i(\xi^-)\,(\overline{\gamma}(\xi))^{-1}$. Then, if node ξ occurs as a result of history $F_t = (\xi_0, \xi_1, \ldots, \xi_{t-1}, \xi)$, we have

$$\mu^i(\xi) \leq \Big[\Big(\prod_{\eta\in F_t\setminus\{\xi_0\}} (\overline{\gamma}(\eta))^{-1}\Big)\, \overline{\mu}^i(\xi_0)\Big] \equiv \tilde{\mu}^i(\xi),$$

and, therefore,

$$\frac{p(\xi,g)}{\sum_{\kappa\in S(\xi,j)} p(\xi,\kappa)} \leq \frac{\tilde{\mu}^i(\xi^-)}{a(\xi,g) \sum_{\kappa\in S(\xi,g)} p(\xi,\kappa)} \tag{33}$$

Equations (32) for case (a) and (33) for case (b) establish the first claim in item (3).

(c) In any case, the last claim of item (3) follows, since

$$\sum_{g\in S(\xi,j)} p(\xi,g) + \sum_{g\notin S(\xi,j)} p(\xi,g) + \sum_{j\in J(\xi)} q_j(\xi) = 1.$$

Part II. Proof of Proposition 5.3.

Lemma 7.1 *Under Assumption [W], an allocation $(x,\theta,\varphi,\Delta)$ which satisfies the conditions of Definition 5.1 is bounded.*

Proof. Following the same idea as in Araujo et al. (2002), one gets:

$$x^i(\xi,g) \le W\mathcal{I}\sum_{n=0}^{t} (\overline{Y}^T G)^n := \varsigma^T < +\infty,\ \forall g\in G, \tag{34}$$

$$\varphi^i_j(\xi) \le \frac{\varsigma^T}{c^j(\xi)} := \alpha^T(\xi) < +\infty,\ \forall j\in J(\xi), \tag{35}$$

$$\theta^i_j(\xi) \le \alpha^T(\xi) < +\infty,\ \forall j\in J(\xi), \tag{36}$$

where $\overline{Y}^T := \max\{(Y(\xi))_{g,g'}, (\xi,g,g')\in D^T\times G\times G\}$ and $c^j(\xi)>0$ was already defined in Remark 2.1.
For each node $\xi\in D^T$, let us define $\chi^T(\xi) = \max\{\varsigma^T(\xi),\alpha^T(\xi)\}$ and $\chi^T = \max_{\xi\in D^T}\chi^T(\xi)$.
Now, for each $i\in I$, let us define:

$$B^{iT}(p,q,K,\chi) = \left\{ (x,\theta,\varphi,\Delta)\in B^{iT}(p,q,K) \;\middle|\; \begin{array}{l} x^i(\xi,g)\le 2\chi^T, \\ \theta^i_j(\xi)\le 2\chi^T, \\ \varphi^i_j(\xi)\le 2\chi^T, \end{array} \right\}$$

Let $\mathcal{E}^T(\chi)$ be the compactified economy which has the same characteristics as $\mathcal{E}^T$ except for the budget constraints which are now defined by the sets $B^{iT}(p,q,K,\chi)$.

Definition 7.1 *A non–trivial equilibrium of the compactified economy $\mathcal{E}^T(\chi)$ is a vector $(\overline{p}^T,\overline{q}^T,\overline{K}^T,(\overline{x}^{iT},\overline{\theta}^{iT},\overline{\varphi}^{iT},\overline{\Delta}^{iT})_{i\in I})$ verifying the conditions (b) and (c) of Definition 5.1 and such that:*

(i') $\forall i\in I$, $(\overline{x}^{iT},\overline{\theta}^{iT},\overline{\varphi}^{iT},\overline{\Delta}^{iT})\in$ Argmax $U^{iT}(x,\theta,\varphi,\Delta)$ over $B^{iT}(p,q,K,\chi)$.

Lemma 7.2 *Under Assumptions [U], each compactified economy $\mathcal{E}^T(\chi)$ has a non–trivial equilibrium $(\overline{p}^T,\overline{q}^T,\overline{K}^T,(\overline{x}^{iT},\overline{\theta}^{iT},\overline{\varphi}^{iT},\overline{\Delta}^{iT})_{i\in I})$.*

Proof. Note that for each $i\in I$, B^{iT} is upper semicontinuous with nonempty closed convex values. For each $(p,q,K)\in\Pi^{T-1}\times[0,1]^{(\sum_{\xi\in D^T}\iota(\xi))}$ and each agent $i\in I$, let us define the set $B'^{iT}(p,q,K,\chi)$ by replacing all the inequalities in $B^{iT}(p,q,K,\chi)$ by strict inequalities.

Claim 7.1 $\forall i\in I$, $\forall (p,q,K)\in\Pi^{T-1}\times[0,1]^{(\sum_{\xi\in D^T}\iota(\xi))}$, $B'^{iT}(p,q,K,\chi)\neq 0$.

Proof. The proof is done by upward induction. Indeed,

- At node $\xi=0$,
  - If $p(\xi_0)\neq 0$, since $\omega^i(\xi_0)\gg 0$, one can choose $x^i(\xi_0)\in X^i(\xi_0)$ such that $p(\xi_0)\cdot(x^i(\xi_0)-\omega^i(\xi_0))<0$. Letting $\theta^i(\xi_0)=\varphi^i(\xi_0)=0$, one gets that the constraints of the period 0 are satisfied strictly.
  - If $p(\xi_0)=0$ (then $q(\xi_0)\neq 0$), one can choose $\theta^i(\xi_0)=0$ and $\varphi^i(\xi_0)\gg 0$ such that $\forall j\in J(\xi_0)$, $\varphi^i_j(\xi_0)<\frac{WI}{c^j(\xi)}$ and $\varphi^i_j(\xi_0)<\frac{p(\xi_0)\omega^i(\xi_0)}{q_j(\xi_0)}$ and the constraints of the period 0 will be satisfied strictly. Note that even in this case, we choose $x^i(\xi_0)\gg 0$ which, as we will see later on, will be helpful for the variable choices of the following period.
- At each $\xi\in{\xi_0}^+$,
  - If $p(\xi)\neq 0$, since $[\omega^i(\xi)+Y(\xi)x^i(\xi_0)]\gg 0$, one can choose $x^i(\xi)\in X^i(\xi)$ such that $p(\xi)\cdot x^i(\xi)<p(\xi)\cdot[\omega^i(\xi)+Y(\xi)x^i(\xi_0)]$. Letting $\theta^i(\xi)=\varphi^i(\xi)=0$, $\Delta^i(\xi)>\varphi^i(\xi_0)\min\{p(\xi)A(\xi),p(\xi)Y(\xi)C(\xi_0)\}$ and $\sum_{j\in J(\xi_0)}\Delta^i_j(\xi)+p(\xi)x^i(\xi)<p(\xi)\Big[\omega^i(\xi)+Y(\xi)\big(x^i(\xi_0)+C(\xi_0)\varphi^i(\xi_0)\big)\Big]$, one gets that the constraints of node ξ are satisfied strictly.
  - If $p(\xi)=0$ (then $q(\xi)\neq 0$), one can choose $\theta^i(\xi)=0$ and $\varphi^i(\xi)>0$ and $\Delta^i(\xi)>0$ such that $\Delta^i(\xi)<q(\xi)\cdot\varphi^i(\xi)$ and the constraints of node ξ will be satisfied strictly. Once again, we choose $x^i(\xi)\gg 0$ which will be helpful for the variable choices at the following period.
- The same tricks can be used until the period $T-1$.
- At node $\xi\in D_T$ (*i.e.* : $t(\xi)=T$). Since $p(\xi)\neq 0$, one can choose $x^i(\xi)\in X^i(\xi)$ such that $p(\xi)\cdot x^i(\xi)<p(\xi)\cdot[\omega^i(\xi)+Y(\xi)x^i(\xi^-)]$.

Claim 7.2 $\forall i\in I, B^{iT}$ *is lower semicontinuous.*

Proof. It follows from the convexity and the non-emptiness of $B'^{iT}(p,q,K,\chi)$ for each $(p,q,K) \in \Pi^T \times [\frac{1}{n},1]^{D^T \times \prod_{\xi \in D^T} J(\xi)}$ that $B^{iT}(p,q,K,\chi) = \overline{B'^{iT}(p,q,K,\chi)}$. The Claim follows from the fact that B'^{iT} is lower semicontinuous (see Moore (2002), ex. 2. page 133).

Let an auxiliary economy $\mathcal{E}_n^T(\chi)$ be a compactified economy with the truncated utility function U^{iT} replaced by the modified utility function U_n^{iT} given by (10).

Definition 7.2 *A non–trivial equilibrium for the auxiliary compactified economy $\mathcal{E}_n^T(\chi)$ is a vector $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})_{i\in I})$ satisfying Definition 7.1 with U_n^{iT} replacing U^{iT} in condition (i').*

Claim 7.3 *There exists a non–trivial equilibrium for the auxiliary compactified economy $\mathcal{E}_n^T(\chi)$.*

Proof. For each agent $i \in I$, let us define the following correspondence:

$$\Psi^{iT}(p,q,K) = \text{Argmax}_{\mathcal{Z}\in B^{iT}(p,q,K,\chi)}\, U_n^{iT}(\mathcal{Z}) \text{ (where } U_n^{iT} \text{ was defined in (10)).}$$

At each node $\xi \in D^T$, we define the correspondence:

$$A^{\xi\, T}\Big((x(\eta),\theta(\eta),\varphi(\eta),\Delta(\eta))_{\substack{\eta=\xi,\xi^i\\ i\in I}}\Big) =$$
$$\text{Argmax}_{(p(\xi),q(\xi))\in\Delta^{G+\iota(\xi)-1}}\Big[p(\xi)\cdot\sum_{i\in I}\Big((x^i(\xi)+C(\xi)\varphi^i(\xi)) - (\omega^i(\xi)+Y(\xi)(C(\xi^-)\varphi(\xi^-)$$
$$+x^i(\xi^-)))\Big) + q(\xi)\cdot\sum_{i\in I}(\theta^i(\xi)-\varphi^i(\xi))\Big],$$

Moreover, we define the following correspondence:

$$L^T\Big((x,\theta,\varphi,\Delta)\Big) = \sum_{i\in I}\Big(\prod_{\xi\in D^T}\prod_{g\in G} h(x^i(\xi,g)) \prod_{j\in J(\xi)} h(\theta_j^i(\xi))\, h(\varphi_j^i(\xi))\Big), \qquad (37)$$

where $h(\mathcal{Z}) = 1 - \frac{\max\{0,\mathcal{Z}-\chi^T\}}{\chi^T}$.

L^T is a sum of products of upper hemicontinuous, compact valued and singleton valued correspondences, then it is upper hemicontinuous, compact valued and singleton valued also. Notice that, if for each $i \in I$ there exists $\xi \in D^T$ such that $x^i(\xi,g) = 2\chi^T$ or $\theta_j^i(\xi) = 2\chi^T$ or $\varphi_j^i(\xi) = 2\chi^T$ for some $g \in G$ or some $j \in J(\xi)$, then $r = L^T\Big((x,\theta,\varphi,\Delta)\Big) = 0$. Otherwise, $L^T\Big((x,\theta,\varphi,\Delta)\Big) > 0$.

Finally, for each node $\xi \in D^T \setminus \{\xi_0\}$, for each asset $j \in J(\xi^-)$, let us define the correspondence:

$$\mathcal{C}_{j,\xi}^T\Big((\varphi_j^i(\xi),\Delta_j^i(\xi)),\, i\in I\Big) =$$
$$\text{Argmin}_{K_j(\xi)\in[\min\{\epsilon_j(\xi),r\},1]}\,\Big[\Big(K_j(\xi)p(\xi)A^j(\xi)\sum_{i\in I}\varphi_j^i(\xi^-) - \sum_{i\in I}\Delta_j^i(\xi)\Big)^2\Big]$$

where, as in Remark 2.1, $\epsilon_j(\xi) = \min\{1, \frac{y(\xi)c^j(\xi^-)}{A^j(\xi)}\,\delta_j(\xi)\}$, and, $\delta_j(\xi)$ is the positive lower bound for $\sum_{\kappa\in S(\xi,j)} p(\xi,g)$ given by item (3) of Corollary 5.1.

The correspondences Ψ^{iT} and $A^{\xi\,T}$ are standard correspondences giving us the maximizers of consumers and of the auctioneers, respectively. The singleton–valued correspondence L^T takes value 0 if some consumer has chosen a quantity greater than χ^T for either the consumption of a good, or the purchase or the sale of some asset, at some node. Otherwise, L^T takes value 1.

For $r = L^T\Big((x,\theta,\varphi,\Delta)\Big)$ and $(x^i,\theta^i,\varphi^i,\Delta^i) \in \Psi^{iT}(p,q,K)$, the constraint set of the correspondence $\mathcal{C}_{j,\xi}^T$ has the following characterization:

If all consumers have chosen for some good, asset purchase or asset sale, at some node, a quantity $2\chi^T$, we have $r = 0$ and $K_j(\xi)$ can be chosen in $[0,1]$. Otherwise, we know that there exists $i \in I$ such that $x^i(\xi,g)$, $\theta_j^i(\xi)$, $\varphi_j^i(\xi)$ are less than $2\chi^T$, for all (ξ,g,j), and the results reported in Proposition 5.2 and its corollary apply (since this consumer i has zero shadow prices for the constraints $x^i(\xi,g) \le 2\chi^T$, $\varphi_j^i(\xi) \le 2\chi^T$, $\theta_j^i(\xi) \le 2\chi^T$). In this case, the compactification of the budget sets was not binding for some consumer and $\delta_j(\xi)$ is actually a positive lower bound for $\sum_{\kappa\in S(\xi,j)} p(\xi,g)$.

In this case, by Remark 2.1, for $\sum_{i\in I}\varphi_j^i(\xi) > 0$, we know that for each $\eta \in \xi^+$, $\text{Argmin}_{K^j(\eta)}\{K^j(\eta)p(\eta)A^j(\eta)\sum_{i\in I}\varphi_j^i(\xi) - \sum_{i\in I}\Delta_j^i(\xi)\}$ on the set $[0,1]$ is the same as the minimizer obtained on $[\min\{\epsilon_j(\xi),r\},1] \subset (\neq)\ [0,1]$, which is the constraint set in this case (as $\min\{\epsilon_j(\xi),r\} \le \epsilon_j(\xi)$ and $r > 0$). For $\sum_{i\in I}\varphi_j^i(\xi) = 0$, this choice of the constraint set will ensure non–triviality of equilibrium.

It is easy to see that Ψ^{iT}, $A^{\xi\,T}$, L^T and $\mathcal{C}_{j,\xi}^T$ are upper hemicontinuous, compact and convex valued correspondences. Define the product correspondence Γ as follows:

$$\Gamma(p,q,x,\theta,\varphi,\Delta,r,K) = \Big(\prod_{\xi\in D^T} A^{\xi\,T}(x_\eta,\theta_\eta,\varphi_\eta,\Delta_\eta)_{\eta=\xi,\xi^i}\Big)\times\Big(\prod_{i\in I}\Psi^{iT}(p,q,K)\Big)$$
$$\times\ \Big(L^T(x,\theta,\varphi,\Delta)\Big)\times\Big(\prod_{\substack{\xi\in D^T\\ j\in J(\xi)}}\mathcal{C}_{j,\xi}^T\Big).$$

By Kakutani theorem, Γ has a fixed point $\Big((\overline{p}^T, \overline{q}^T, \overline{K}^T), (\overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T), \overline{r}^T\Big)$ in $\Big(\prod_{\xi \in D^T} \Delta^{G+\iota(\xi)-1} \times [0,1]^{\iota(\xi)} \times \mathbb{R}^{(G+\iota(\xi))\mathcal{I}}\Big) \times [0, \mathcal{I}]$.

Let us show that $(\overline{p}^T, \overline{q}^T, \overline{K}^T), (\overline{x}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T), \overline{r}^T\Big)$ is an equilibrium for the auxiliary truncated compactified economy $\mathcal{E}_n^T(\chi)$. Next, we will show that it is also an equilibrium for the truncated economy, $\mathcal{E}^T$, and that it is non–trivial.

First, $(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$ maximizes U_n^{iT} on the set $B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T, \chi)$. Secondly, item (ii) of Definition 2.2 holds at $(\overline{x}^{iT}, \overline{\varphi}^{iT})$, that is commodity markets at node ξ_0 clear. In fact, by definition of the correspondence $A^{\xi_0\, T}$, excess demand is ruled out and, if an excess of supply occurred for some $g \in G$, then $\overline{p}^T(\xi, g) = 0$, implying, by monotonicity of the preferences, that all the agents would consume $2\,\chi^T$ of this good, more than its aggregate endowment. Third, regarding Condition (iv) of item (b), Definition 5.1, we see that excess demand for $j \in J(\xi_0)$ is also ruled out (by definition of $A^{\xi_0\, T}$).

Then, for $\xi \in \xi_0$, we have: $\sum_{j \in J(\xi_0)} \Big[\overline{K}_j^T(\xi)\overline{p}^T(\xi)A^j(\xi)\sum_{i\in I}\overline{\theta}_j^{iT}(\xi_0) - \sum_{i \in I}\overline{\Delta}_j^{iT}(\xi)\Big] \leq$ $\sum_{j \in J(\xi_0)} \Big[\overline{K}_j^T(\xi)\overline{p}^T(\xi)A^j(\xi)\sum_{i\in I}\overline{\varphi}_j^{iT}(\xi_0) - \sum_{i \in I}\overline{\Delta}_j^{iT}(\xi)\Big]$, which is zero, clearly when $r = 0$ and, otherwise, for $r > 0$, it is also zero, since, for $\sum_{i\in I}\overline{\varphi}_j^{iT}(\xi) > 0$, it is zero for $\overline{K}_j^T(\xi) \geq \epsilon_j(\xi)$, and for $\sum_{i \in I}\overline{\varphi}_j^{iT}(\xi) = 0$, it is trivially zero (as $\overline{\Delta}_j^{iT}(\xi)$ becomes zero also, for each i).

Hence, by definition of $A^{\xi\, T}$, for $\xi \in {\xi_0}^+$, we can again rule out excess demand of commodities and asset traded at ξ. By monotonicity of preferences, excess supply in commodity markets is also eliminated. We then proceed to $\eta \in \xi^+$ and repeat the argument till we reach date T.

Having ruled out excess of demand for all goods and all assets, at any node $\xi \in D^T$, it follows that $\overline{r}^T = 1$ and, result (3) of Corollary 5.1 holds (as $x^i(\xi, g)$, $\theta_j^i(\xi)$, $\varphi_j^i(\xi) \leq \chi^T < 2\chi^T$, $\forall i \in I$, $\forall g \in G$, $\forall j \in J(\xi)$). Then, $\overline{K}_j^T(\xi) \geq \epsilon_j(\xi) > 0$, whether asset $j \in J(\xi)$ is traded or not. This fact has two consequences. First, excess supply in asset markets is ruled out, as it would require $\overline{q}_j^T(\xi) = 0$, but for $\overline{K}_j^T(\eta) > 0$ and $\overline{p}^T(\eta) > 0$, $\forall \eta \in \xi^+$, $\overline{\theta}_j^{iT}(\xi)$ would be equal to $2\chi^T$, for each i, contradicting absence of excess demand. Secondly, the equilibrium we just established for the auxiliary truncated compactified economy, $\tilde{\mathcal{E}}_n^T$ (with modified penalties $\tilde{\lambda}_j^{i,n}(\xi)$) , is non-trivial, as, even in the absence of asset trades we have $\overline{K}_j^T(\xi) > 0$, $\forall \xi \in D^T$, $\forall j \in J(\xi)$. □

Claim 7.4 *The non–trivial equilibrium* $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})_{i\in I})$ *for the auxiliary economy* $\mathcal{E}_n^T(\chi)$ *is also a non–trivial equilibrium for the compactified economy* $\mathcal{E}^T(\chi)$.

Proof. In fact, in an auxiliary economy, $\tilde{\lambda}_j^{i,n}(\xi) = \lambda_j^i(\xi)\frac{1}{\max\{1/n,\, p(\xi)b(\xi)\}}$, and by Proposition 5.2, item (iii), $\sum_{g\in G} p(\xi, g)$ has a positive lower bound, for each ξ and independent of n. Then, $\tilde{\lambda}_j^{i,n}(\xi) = \frac{\lambda_j^i(\xi)}{p(\xi)\,b(\xi)}$, for n large enough. This completes the proof of Lemma 7.2. □

To complete the proof of Proposition 5.3, we just need the following standard lemma:

Lemma 7.3 *The non–trivial equilibrium for the compactified economy* $\mathcal{E}^T(\chi)$ *is a non–trivial equilibrium for the truncated economy* $\mathcal{E}^T$.

Proof. Let $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})_{i\in I})$ be an equilibrium of $\mathcal{E}^T(\chi)$. In particular, this collection satisfies the feasibility conditions stated in item (b) of Definition 5.1. Moreover, $\forall i \in I$, $(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT}) \in \text{Argmax } U^{iT}(x, \theta, \varphi, \Delta)$ over $B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T, \chi)$. Then, it remains to prove the optimality of $(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$ in $B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T)$. By contraposition, let us assume that there exists $i \in I$ and $(x^i, \theta^i, \varphi^i, \Delta^i) \in B^{iT}(\overline{p}^T, \overline{q}, \overline{K})$ such that $U^{iT}(x^i, \theta^i, \varphi^i, \Delta^i) > U^{iT}(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$. Since $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})_{i\in I})$ is an equilibrium of $\mathcal{E}^T(\chi)$, it follows from conditions (34)–(36) that $(\overline{x}^{iT}(\xi), \overline{\theta}^{iT}(\xi), \overline{\varphi}^{iT}(\xi), \overline{\Delta}^{iT}(\xi))$ is an interior point of the compactified set $B^{iT}(\overline{p}^T, \overline{q}, \overline{K}, \chi)$. Since there is a finite number of nodes in the truncated economies, one can choose $\beta \in (0, 1)$, β close to zero, such that:

$$[(1-\beta)(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT}) + \beta(x^i, \theta^i, \varphi^i, \Delta^i)] \in B^{iT}(\overline{p}^T, \overline{q}, \overline{K}, \chi)$$

and

$$U^i([(1-\sigma)(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT}) + \sigma(x^i, \theta^i, \varphi^i, \Delta^i)] > U^i(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT}),$$

which contradicts the optimality of $(\overline{x}^{iT}, \overline{\theta}^{iT}, \overline{\varphi}^{iT}, \overline{\Delta}^{iT})$ in $B^{iT}(\overline{p}^T, \overline{q}^T, \overline{K}^T, \chi)$ proved in Lemma 7.2. □

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