AMERICAN UNIVERSITY OF BEIRUT
DEPARTMENT OF MATHEMATICS
ALGEBRA COMPREHENSIVE EXAM
FALL 2014

1. (8 pts) Determine the values of k for which the system having the following augmented matrix is consistent.

$$\left[\begin{array}{ccc|c} 4 & 0 & k & 1 \\ 0 & k-2 & 1 & 3 \\ 0 & 0 & k-1 & 2 \\ 0 & 0 & 0 & k^2-4 \end{array}\right]$$

2. (14 pts) Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A.

(c) Deduce that A is diagonalizable.

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

3. (8 pts) Show that if a system of linear equations has two distinct solutions then it has infinitely many solutions.

4. (12 pts) Let $T : V \to W$ be a linear transformation between vector spaces and let $\{v_1, \ldots, v_n\}$ be a linearly independent subset of V.

a) Show that if T is one-to-one then $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.

b) Let $V = W = \mathbb{R}^3$. Give an example of a linearly independent subset $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ and a nontrivial linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

5. (8 pts) Let V be an inner product space and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of nonzero vectors. Show that S is linearly independent.

6. (10 pts) Let G be a group and let a and b be elements of G of finite order. Show that if the orders of a and b are relatively prime then $< a > \cap < b >= \{e\}$.

7. (10 pts) Let G be a group and let $\phi : G \to G$ be given by $\phi(g) = g^{-1}$. Show that ϕ is an isomorphism if and only if G is abelian.

8. (10 pts) Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

9. (10 pts) Let R be a commutative ring with unity and let F be a field. Suppose that $f : R \to F$ is a surjective ring homomorphism. Show that $\mathrm{Ker}(f)$ is a maximal ideal of R.

10. (10 pts) Let D be an integral domain and let $p \in D$. Show that if p is prime then p is irreducible.