AMERICAN UNIVERSITY OF BEIRUT
Algebra Comprehensive Exam

Fall 2015

1. (10 points) Consider the system with augmented matrix $\left[\begin{array}{ccc|c} 1 & k & k+3 & 4 \\ k & -k & 2k & 4 \\ k & -k & 3k-2 & k^2 \end{array}\right]$.

Find all values of k for which this system has

a) a unique solution b) no solution c) infinitely many solutions.

2. (10 points) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

a) Find the eigenvalues of A and deduce that A is diagonalizable.

b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

3. (10 points) Let $\mathcal{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be given by $T(p(x)) = xp'(x) + x^2 p(0)$. Prove or disprove:

a) T is a linear transformation. b) T is one-to-one. c) T is onto.

4. (10 points) Let A be an $n \times n$ matrix with $A^2 = 0$. Show that the column space of A is a subspace of the null space of A. Deduce that $\text{Rank}(A) \leq n/2$.

5. (10 points) Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of an inner product space V. Show that S is linearly independent.

6. (14 points) Let W be a subspace of a finite-dimensional real vector space V and let R be the ring of all linear transformations $T : V \to V$ with the operations of addition and composition of transformations. Let $I = \{T \in R : W \subset N(T)\}$, where $N(T) =$ the null space of $T =$ the kernel of T. Determine whether each of the following statements is true or false. If the statement is true, prove it. If the statement is false, give a counterexample.

a) I is a right ideal of R. b) I is a left ideal of R.

7. (6 points) Let G be the group $\mathbb{Z} \times (\mathbb{Z}/12\mathbb{Z})$ and let H be the set of all elements of G having infinite order, together with the identity of G. Show that H is not a subgroup of G.

8. (10 points) Find all group homomorphisms $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_5$.

9. (10 points) Let H be a normal subgroup of a finite group G and let $n = |G|/|H|$. Show that $a^n \in H$ for all $a \in G$.

10. (10 points) Let R be a commutative ring with unity. Show that every maximal ideal of R is a prime ideal.