Algebra

Comprehensive Examination **Aug 2016**

Time allowed: 90 minutes

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1. Let T: V→W be a linear transformation from a vector space V to a vector space W. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a set of vectors in V such that $\{T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)\}$ is a linearly independent subset of W, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly independent in V.
[10 points]

2. **Prove** or **Disprove**:

   (a) The matrix $A=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *is not diagonalizable*
   [5 points]

   (b) An orthogonal set of vectors in an inner product space V is linearly independent in V.
   [5 points]

3. Let V be a vector space of dimension 4, and let W be a subspace of V with **basis** $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ **.** Show that there exists a linear operator T on V such that $T(\mathbf{w})=\mathbf{w}$ for all $\mathbf{w} \in W$ but $T(\mathbf{u}) \neq \mathbf{u}$ for some $\mathbf{u}$ in V
[10 points]

4. Let $T: V \to V$ be a linear operator on a vector space V such that
$$T^2 = T \quad \left(\text{that is} \quad T(T(v) = T(v) \ \forall v \in V\right)$$
Show that
$$V = N(T) + R(T), \text{ where } N(T) \text{ is the Nullspace of } T \text{ and } R(T) \text{ is the Range of } T$$
[Hint: Use the vector $(v - T(v))$]
[10 points]

5. Suppose that φ: R→S is a ring homomorphism such that R has identity 1. Prove that if φ is surjective (onto), then φ(1) is the identity of S.
[10 points]

6. Let H be a normal subgroup of a group G such that $O(G/H) = n$. Prove that $a^n \in H$ for every $a \in G$.
[10 points]

7. Let $\varphi: G \rightarrow H$ be a group homomorphism such that H is abelian. Prove that if N is a subgroup of G containing Ker φ, then N is normal in G.

   (Hint: consider $\varphi(gng^{-1}n^{-1})$,$\forall g \in G, \forall n \in N$)

[10 points].

8. Let R be a commutative ring with identity such that for every element $a \in R$, *there exists* $a' \in R$ *such that* $aa'a = a$. Prove that every prime ideal of R is maximal.

[10 points]

9. Answer **TRUE** or **FALSE** only (**2 points for each correct answer):**

1. Let A be a 3×3 matrix such that $A^2 + A + 2I = 0$, then A is invertible.
2. Similar matrices have the same determinant
3. If λ is any eigenvalue of an invertible nxn matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}
4. S_{2x2} ={symmetric 2x2 matrices} is a subspace of M_{2X2} isomorphic to the vector space P_2
5. If A is a 2x5 matrix with orthogonal nonzero row vectors then nullity(A)=3
6. Any orthogonal set of 4 nonzero vectors in R^4 forms a basis for R^4
7. The alternating group A_n of all even permutations in S_n (n>1) is a normal subgroup of S_n.
8. If D is an integral domain such that 4a=0 for some a≠0 in D, then D has finite Characteristic=4.
9. If $\varphi: Z_7 \rightarrow H$ is a nontrivial ring homomorphism, then φ is one-to-one.
10. Let G be a group of order 20, then G has a normal subgroup of order 5

[20 points]