American University of Beirut
Algebra Comprehensive Exam
Fall 2017
Time allowed: 2h

Problem 1. Let G be a cyclic group of order 100 with generator g. Let $h = g^3$.

(a) Show that h is also a generator of G.
(b) Write g^{16} as a power of h. (Can be solved even if you did not do part (a).)

Problem 2.
Let $\sigma = (12345), \tau = (123)(45)$ be elements of the symmetric group S_5.

(a) Which of σ, τ, $\tau \circ \sigma$, and $\tau \circ \sigma \circ \tau^{-1}$ belong to the alternating group A_5?
(b) Find the orders of each of σ, τ, $\tau \circ \sigma$, and $\tau \circ \sigma \circ \tau^{-1}$.

(Parts (a) and (b) are independent of each other.)

Problem 3. Consider the ring $\mathbb{Z}$ of integers, and the subset $I \subset \mathbb{Z}$ consisting of multiples of 5. Thus $I = \{\dots, -5, 0, 5, 10, 15, \dots\}$.

(a) Show that I is an ideal of $\mathbb{Z}$.
(b) Explicitly describe all the cosets of I in $\mathbb{Z}/I$. In other words, list all the cosets, and for each coset, describe all its elements.
(c) Is the quotient ring $\mathbb{Z}/I$ an integral domain? Why or why not?

Problem 4. Consider the following subset of the ring $M_2(\mathbb{R})$ of 2×2 matrices:

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{R} \right\}.$$

(a) Show that S is a subring of $M_2(\mathbb{R})$.
(b) Show that S is not commutative.
(c) Give a homomorphism of *rings* $\phi : S \to \mathbb{R} \times \mathbb{R}$ and describe its kernel.

Problem 5. Let V, W be vector spaces, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for V. Let $T : V \to W$ be a linear transformation.

(a) Show that T is injective (meaning "one-to-one") *if and only if* the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), T(\mathbf{v}_4)$ are linearly independent in W.
(b) Interpret the above result in terms of the Rank-Nullity Theorem (the one relating $\dim \ker T$ and $\dim \operatorname{image} T$).

Problem 6. Find a basis for the kernel $\ker T$ and the image $\operatorname{image} T$ where $T : \mathbb{R}^5 \to \mathbb{R}^3$ is the linear transformation given by

$$T\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 & 2 \\ 2 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}.$$

Problem 7. Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \mid a + b + c + d = 0. \right\}$$

Find an *orthogonal* basis for W. (It is not obligatory to make your basis orthonormal.)

Problem 8. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors for A.
(b) Write the nth power of A as

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Using part (a), give an explicit formula for a_n, b_n, c_n, d_n.