American University of Beirut
Algebra Comprehensive Exam
Fall 2018
Time allowed: 2h

Problem 1. Let G be a group. *We do NOT assume that G is Abelian.*

(a) Let $Z(G) = \{g \in G \mid \forall h \in G,\ gh = hg\}$. Prove that $Z(G)$ is a subgroup of G.

(b) An example: determine $Z(G)$ when G is the symmetric group S_n ($n \in \mathbb{N}$).
Hint: treat the cases $n \leqslant 2$ and $n \geqslant 3$ separately.

(c) We now denote by$\operatorname{Aut}(G)$ the set of automorphisms of G (recall that an automorphism of G is an isomorphism from G to itself). Prove that $\operatorname{Aut}(G)$ is a group under composition.

(d) Let $g \in G$, and consider the map

$$\begin{array}{rcl} \varphi_g : G & \longrightarrow & G \\ h & \longmapsto & ghg^{-1} \end{array} .$$

Prove that $\varphi_g \in \operatorname{Aut}(G)$ for all $g \in G$.

(e) Prove that the map

$$\begin{array}{rcl} \varphi : G & \longrightarrow & \operatorname{Aut}(G) \\ g & \longmapsto & \varphi_g \end{array}$$

is a homomorphism. What is its kernel?

(f) Give an example of group G such that φ is not surjective.

Problem 2.
Let $a, b, c, d \in \mathbb{R}$, and let

$$M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & d \end{pmatrix}.$$

(a) Find necessary and sufficient conditions on a, b, c, d for M to be invertible.

(b) Find necessary and sufficient conditions on a, b, c, d for M to be diagonalizable.

(c) Find necessary and sufficient conditions on a, b, c, d for M to be nilpotent.

Remark: parts (a), (b), and (c) are independent from each other. You must justify your answers.

Problem 3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation whose matrix with respect to the usual basis of $\mathbb{R}^3$ is

$$M = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}.$$

(a) Determine the rank of f and the determinant of f.

(b) Find a basis for $\operatorname{Im} f$ and for $\operatorname{Ker} f$.

(c) Find a basis of $\mathbb{R}^3$ in which the matrix of f has only one non-zero coefficient.

(d) Compute M^n in terms of $n \in \mathbb{N}$.

Problem 4. Let R be a *commutative* ring with unit.

(a) Let I and J be ideals of R, and let $I + J = \{i + j \mid i \in I,\ j \in J\}$. Prove that $I + J$ is an ideal of R.

(b) Let I and J be ideals of R, and let IJ be the subset of R consisting in the elements of the form

$$\sum_{k=1}^{n} i_k j_k$$

where $n \in \mathbb{N}$ and $i_k \in I$, $j_k \in J$ for all k. In other words, IJ is the set of finite sums of products of an element of I by an element of J. Prove that IJ is an ideal of R.

(c) Let I and J be ideals of R. Order the ideals I, $I + J$, $I \cap J$, and IJ by inclusion. Give an example where all the inclusions are strict.

(d) Let I be an ideal of R. We define

$$\operatorname{rad}(I) = \{x \in R \mid \exists n \in \mathbb{N} : x^n \in I\}.$$

Prove that $\operatorname{rad}(I)$ is an ideal of R.
Hint: Use the formula with binomial coefficients for $(x + y)^m$.

(e) An example: in this question only, we take $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$. What is $\operatorname{rad}(I)$?

(f) Let I be an ideal of R. Prove that $\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I)$.

(g) Let I be an ideal of R. Prove that if I is a prime ideal, then $\operatorname{rad}(I) = I$. Is the converse true?

(h) Let I and J be ideals of R. Prove that $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad}(I) \cap \operatorname{rad}(J)$.

(i) Recall that an element $x \in R$ is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $x^n = 0$; in particular 0 is nilpotent. Let $\operatorname{Nil}(R)$ be the set of nilpotent elements in R. Prove that $\operatorname{Nil}(R)$ is an ideal of R.
Hint: Use question (d).

(j) Prove that $R/\operatorname{Nil}(R)$ has no nonzero nilpotent element.

End