Comprehensive AUB - Fall 2019
Duration: 1h30

Exercise 1. Let G be a finite group of order n. Denote by e its identity. Show that $a^n = e$ for every $a \in G$.

Exercise 2. Let H be a normal subgroup of a group G such that H is of index n in G (i.e. $o(G/H) = n$). Prove that $a^n \in H$ for every $a \in G$.

Exercise 3. Let V and W be finite dimensional vector spaces over a field F. Consider a proper subspace U of V and let $v \in V$ such that $v \notin U$. Let $w \in W$. Show that there exists a linear transformation $f : V \to W$ such that $f(u) = 0$ for all $u \in U$ and $f(v) = w$.

Exercise 4. Prove that every homomorphism from a field F to a ring R is either one-to-one (injective) or maps all of F onto $\{0\}$.

Exercise 5. Let $n \in \mathbb{N}$ and $V = \mathrm{P}_n$ be the set of polynomials with real coefficients and with degree $\leq n$. We endow V with its the usual structure of vector space. For every $P \in V$, we denote by P' its derivative. Let $T : V \to V$ defined by $T(P(x)) = P(x) + P'(x)$.

1. Let $S := \mathrm{id}_V - T$. Show that S is a nilpotent operator on V (i.e. $S^m = 0$ for some integer m) using two methods:
   - (a) Using directly the definition of S
   - (b) By writing the matrix of T in a suitable basis of V
2. What are then the eigenvalues of T? Is T diagonalizable?

Exercise 6. Let R be a commutative ring with identity 1 such that for each $a \in R$, there exists $b \in R$ such that $a^2 b = a$. Prove that every prime ideal of R is maximal.

Exercise 7. Let K be a cyclic normal subgroup of a group G. Prove that every subgroup of K is normal in G.