Algebra

Comprehensive Examination   November 26, 2020

Time allowed: 90 minutes

1. Show that If T: $R^2 \longrightarrow R$ is a linear transformation with $T\begin{pmatrix}1\\1\end{pmatrix} = T\begin{pmatrix}2\\1\end{pmatrix} = T\begin{pmatrix}1\\2\end{pmatrix}$, then T is the zero linear transformation. (that is show T(v)=0 for any v $\in R^2$)

   [10 points]

2. Show that if $<\boldsymbol{u}, \boldsymbol{v}> = <\boldsymbol{u}, \boldsymbol{w}>$ for all vectors **u** in an inner product space V, then **v = w.**

   [10 points]

3. Let V be a vector space of dimension 3, and let W be a subspace of V with **basis** {$\mathbf{w_1}$, $\mathbf{w_2}$} **.** Show that there exists a linear operator T on V such that T(**w**)=**0** for all **w**$\in$W and T(**u**)$\neq$ **0** for some vector **u** in V.

   [10 points]

4. Let H be a normal subgroup of a group G such that $O(G/H) = n$. Prove that $a^n \in H$ for every $a \in G$.

   [10 points]

5. Let $\varphi: G \rightarrow H$ be a group homomorphism such that H is abelian. Prove that if N is a subgroup of G containing Ker φ, then N is normal in G.
   (Hint: consider $\varphi(gng^{-1}n^{-1})$, $\forall g \in G, \forall n \in N$)

   [10 points].

6. Let R be a commutative ring with identity such that for every element $a \in R$, *there exists* $a' \in R$ *such that* $aa'a = a$. Prove that every prime ideal of R maximal.

   [12 points]

7. Let R and S be rings with identity elements 1_R and 1_S respectively. Let $\varphi : R \rightarrow S$ be a ring homomorphism. Suppose that there is an invertible element a in R such that $\varphi(a)$ is invertible in S. Prove that $\varphi(1_R) = 1_S$.

   [12 points]

Answer **TRUE** or **FALSE** only [**2 points for each correct answer, NO PENALTY]**

1. If A and B are two nxn matrices such that AB = 3I , then the column space Col(A) = R^n.

2. A subspace W of a vector space V is linearly independent.

3. Let A be an nxn matrix such that $A^2 = I$, then 0 is an eigenvalue of A .

4. Let A be a 3x3 matrix with eigenvalues 1, 2, and 3, then rank(A)<3

5. S ={All symmetric 2x2 matrices} is a subspace of M_{2X2} Isomorphic to the vector space P_2

6. Any orthogonal set of 4 vectors in P_3 forms a basis for P_3

7. If A is a 4x6 matrix with rank A = 3 then dim N(A)= 1 (N(A) denotes the Nullspace of A)

8. Let V be a finite-dimensional vector space and let T: V→V be a linear operator on V. If T is one-to-one, then T is onto.

9. If $\varphi : Z_5 \to H$ is a nontrivial group homomorphism, then Kerφ={e} .

10. The matrices A=$\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$ and B=$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ are similar.

11. Let G be a group of order m, and let a ∈G. Then a^m = e .

12. The permutation group S_3 has a subgroup of order 4

13. If G is a group such that a^2=e for all a∈G. then G is abelian

[26 points]