American University of Beirut, Mathematics Comprehensive Exam, Spring 2013

Problem 1. Let G be a cyclic group of order 10, written multiplicatively. Let $x \in G$ be a generator.
a) List all the generators of G.
b) Give an injective (i.e., one-to-one) homomorphism $\phi : G \to \mathbf{C}^*$; here $\mathbf{C}^*$ is the multiplicative group of nonzero complex numbers.
c) Give a nontrivial noninjective homomorphism $\psi : G \to \mathbf{C}^*$. (Nontrivial means that $\psi(g)$ is not always equal to 1.)

Problem 2. Let R be a possibly noncommutative ring.
a) Define what it means for a subset $I \subset R$ to be an ideal.
b) Show that if I and J are ideals of R, then $I \cap J$ is also an ideal.
c) In the ring $\mathbf{Z}$, find the intersection of the ideals $30\mathbf{Z}$ and $33\mathbf{Z}$.

Problem 3. Recall that the alternating group A_4 has order $|A_4| = 12$.
a) List all the elements of A_4, and list their orders.
b) Find a subgroup $H \subset A_4$ of order 4. Be sure to explain why it is a subgroup.
c) Briefly sketch why H is a normal subgroup of A_4.

Problem 4. Let R be a commutative ring (with 1), let K be a field, and let $\phi : R \to K$ be a homomorphism of rings. (We assume that ϕ respects the multiplicative units: $\phi(1_R) = 1_K$.)
a) Show that $\ker \phi$ is a prime ideal of R.
b) When is $\ker \phi$ a maximal ideal of R?
c) Let $R = \mathbf{Q}[x]$ be the ring of polynomials with rational coefficients, and let $K = \mathbf{C}$. Define $\phi : R \to \mathbf{C}$ to be the homomorphism of "evaluation at $\sqrt{2}$":

$$\phi(f(x)) = f(\sqrt{2}).$$

Describe $\ker \phi$ and $\operatorname{image} \phi$. (You may use without proof the fact that all ideals in $\mathbf{Q}[x]$ are principal.)

Problem 5. Define a linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^4$ by

$$T\left(\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}\right) = \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 3 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 3 & 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}.$$

Find a basis for each of $\ker T$ and $\operatorname{image} T$.

Problem 6. Let V be an inner product space, and let $e_1, \ldots, e_n \in V$ be orthonormal. This means that

$$\langle e_i, e_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Caution: we do not assume that $\{e_1, \ldots, e_n\}$ is a basis for V.
a) Show that $e_1, \ldots, e_n$ are linearly independent.
b) Let $W = \operatorname{span}\{e_1, \ldots, e_n\} \subset V$. For $v \in V$, let $v' \in W$ be the orthogonal projection of v onto W. Your job is to give (with justification) a formula for v' in terms of v, the $\{e_i\}$, and inner products.

Problem 7. Let $V = \mathcal{P}_3$ be the vector space of polynomials of degree ≤ 3, with coefficients in $\mathbf{R}$. Define a linear transformation $T : V \to V$ by

$$Tf = xf' - f''.$$

For example, $T(x^3 + 4x^2) = x(3x^2 + 8x) - (6x + 8) = 3x^3 + 8x^2 - 6x - 8$.
a) Write down the matrix for T with respect to the basis $\{1, x, x^2, x^3\}$ of V.
b) Find the eigenvalues and eigenvectors of T.

Problem 8. Let V and W be finite-dimensional vector spaces over $\mathbf{R}$, and let $T : V \to W$ be an injective linear transformation.
a) Show that $\dim V \leq \dim W$.
b) Show that there exists a linear transformation $S : W \to V$ such that $ST = \mathrm{id}_V$.
c) If $\dim V < \dim W$, show that for **every** $S : W \to V$, we have $TS \neq \mathrm{id}_W$.
(Here $\mathrm{id}_V : V \to V$ and $\mathrm{id}_W : W \to W$ are the identity transformations. Be sure to distinguish between $ST = S \circ T$ and $TS = T \circ S$.)