Algebra
Comprehensive Examination April 2014

1. Let T: V$\rightarrow$W be a linear transformation such that T is one-to-one. Prove that if $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ is a linearly independent set of vectors in V, then $\{T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})\}$ is a linearly independent subset of W.

2. (a) Let A be a 3$\times$3 matrix such that $AA^T + A^2 + 2I = 0$. Show that A is invertible and Find A^{-1}.
   (b) Show that similar n$\times$n matrices have the same determinant.

3. Let V be a vector space of dimension 4, and let **u** be a nonzero vector of V**.** Show that there exists a subspace W of **V** such that dim W=3 and $\mathbf{u} \notin W$.

4. Show that an orthogonal set of nonzero vectors in an inner product space is linearly independent.

5. Let G be a group of order 75. Show that G has a normal subgroup of order 25.

6. Prove that every subgroup of a cyclic group is cyclic.

7. Prove or Disprove:

   (a) If φ: R$\rightarrow$S is a ring homomorphism such that R has identity 1, then $\varphi(1)$ is the identity of S.
   (b) If G is a group such that G has only one automorphism (the identity automorphism), then G is abelian.

8. Let R be a commutative ring with identity such that for every element $a \in R$, *there exists a unique element* $b \in R$ *such that* $aba = a$. Prove that
   (a) R has no zero divisors
   (b) R is a field.