American University of Beirut, Mathematics Department
Algebra Comprehensive Exam, April 2015. Time: 90 minutes

1. Let V and W be vector spaces (over $\mathbf{R}$), and let $T : V \to W$ be a linear transformation.

a) Carefully define the kernel (also called nullspace) $\operatorname{Ker} T$, and the image $\operatorname{Image} T$, and show that they are subspaces of V and W, respectively.

b) Let V and W be finite-dimensional vector spaces, and suppose that $T : V \to W$ is a linear transformation. Prove that T is injective (i.e., one-to-one) if and only if there exists a linear transformation $S : W \to V$ with $S \circ T = id_V$ ("id_V" is the identity on V).

c) Let V be a 3-dimensional vector space with basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3\}$, and let $T : V \to V$ be given by

$$T(\vec{\mathbf{e}}_1) = 2\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3, \qquad T(\vec{\mathbf{e}}_2) = 2\vec{\mathbf{e}}_1 + 3\vec{\mathbf{e}}_2 + 2\vec{\mathbf{e}}_3, \qquad T(\vec{\mathbf{e}}_3) = -\vec{\mathbf{e}}_1 - \vec{\mathbf{e}}_2.$$

Write the matrix of T with respect to the basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3\}$, and compute its characteristic polynomial.

d) With repect to the above transformation T from part (c), find a new basis $\{\vec{\mathbf{f}}_1, \vec{\mathbf{f}}_2, \vec{\mathbf{f}}_3\}$ with respect to which the matrix for T becomes **diagonal**.

2. Let G be a finite group of order $|G|$.

a) Show that if $|G| = p$ is prime, then G is cyclic.

b) If $|G| = 10$ and G is abelian, show that G is cyclic.

c) Let D_5 be the dihedral group, i.e., group of symmetries of a regular pentagon. Show that $|D_5| = 10$ and that D_5 is not abelian. (Note: you can, if you wish, view D_5 as a subgroup of the symmetric group S_5; this may help with the notation.)

d) List the subgroups of D_5 and indicate which ones are normal subgroups.

3. Let R be a ring with identity.

a) Define the terms **ideal** and **left ideal**.

b) Let $a, b \in R$. Show that the set $I = \{xa + yb \mid x, y \in R\}$ is a left ideal containing a and b, and that it is the smallest such left ideal. We write $I = \langle a, b\rangle$.

c) Show that for all $a, b, q \in R$, one has the equality of left ideals $\langle a, b\rangle = \langle b, a - qb\rangle$.

d) Let $R = \mathbf{Z}$. Show that $\langle 100, 70\rangle = \langle 10\rangle$, where the notation $\langle a\rangle$ is defined by $\langle a\rangle = \{xa \mid x \in \mathbf{Z}\}$.