American University of Beirut
Department of Mathematics
Algebra Comprehensive Exam

1. (12 points) Let $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ -2 & -6 & 4 & 8 \\ 1 & 3 & 0 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$.

(a) Find a basis for the row space of A, the column space of A and the null space (kernel) of A .

(b) Find all values of k for which b belongs to the column space of A.

2. (6 points) Let $T : \{\sum_{i=0}^{2} a_i x^i : a_i \in \mathbb{R}\} \to \mathbb{R}^2$ be a linear transformation. If $T(3x^2 - 2x + 1) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ and $T(3x^2 + 4x + 3) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, find $T(9x^2 - 24x - 3)$.

3. (30 points) Determine whether each of the following statements is true or false. If a statement is true, prove it, otherwise give a counterexample.

(a) Let V and W be vector spaces over $\mathbb{R}$. If X and Y are subspaces of V and $T : V \to W$ is a linear transformation, then $U = \{T(x) - T(y) : x \in X, y \in Y\}$ is a subspace of W.

(b) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$ and A is an $n \times n$ non-invertible matrix, then $Av_1, Av_2, \ldots, Av_n$ are linearly dependent.

(c) Let A be a 2×2 invertible matrix and let v_1, v_2 be two eigenvectors of A. If Av_1 and Av_2 are orthogonal then A is diagonalizable.

4. (10 points) Let K be a cyclic normal subgoup of a group G. Show that every subgroup of K is normal in G.

5. (10 points) Let R be an integral domain. Show that every prime in R is irreducible.

6. (10 points) Let X be a nonempty subset of a commutative ring R and let $A(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. Show that $A(X)$ is an ideal of R.

7. (10 points) Let R be a ring with unity 1 and let ϕ be a nontrivial ring homomorphism mapping R into an integral domain R'. Show that $\phi(1)$ is the unity of R'.

8. (12 points) Give an example of a nontrivial homomorphism ϕ for the given groups, if an example exists. If no such homomorphism exists, explain why that is so.

a) $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{20}$

b) $\phi : \mathbb{Z}_{28} \to \mathbb{Z}_{15}$

c) $\phi : \mathbb{Z} \to \mathbb{Z}_4$

d) $\phi : \mathbb{Z}_4 \to \mathbb{Z}$