Comprehensive exam

American University of Beyrouth
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Exercise 1. All your answers should be justified briefly.

1. Let $E \subseteq \mathbb{R}^2$ be the parabola of cartesian equation $y = x^2$. Is E a subspace of the real vector space $\mathbb{R}^2$?

2. Give an example of two **proper** subspaces E and F of the real vector space $\mathbb{R}^3$ such that $\mathbb{R}^3 = E + F$ but $\mathbb{R}^3$ is not equal to the direct sum of E and F.

Exercise 2. Endow $\mathbb{R}^4$ with its natural structure of inner product space. Let $E \subseteq \mathbb{R}^4$ be the plane spanned by the vectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (0, 1, 1, 1)$. Denote by $E^\perp$ its orthogonal in $\mathbb{R}^4$. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the orthogonal projection onto E.

1. Find $v_3, v_4 \in \mathbb{R}^4$ such that $E^\perp$ is spanned by v_3 and v_4.

2. Determine the matrix of f in the basis (v_1, v_2, v_3, v_4) of $\mathbb{R}^4$.

3. Deduce the matrix of f in the canonical basis of $\mathbb{R}^4$.

Exercise 3. Let $n \in \mathbb{N}$, and let

$$A_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

be the matrix of size $n \times n$ whose entries are all 1.

1. Compute the image of A_n and the rank of A_n.

2. Prove that $A_n^2 = nA_n$.

3. Deduce that the only possible eigenvalues of A_n are $\lambda_n = 0$ and $\lambda_n = n$.

4. Prove that the dimension of the eigenspace of A_n for $\lambda_n = 0$ is $n - 1$, and that the dimension of the eigenspace for $\lambda_n = n$ is 1, and give bases of these eigenspaces.

5. What is the characteristic polynomial of A_n ?

Exercise 4. Let S_7 be the symmetric group on $\{1, 2, \cdots, 7\}$, and let $\sigma \in S_7$ be such that $\sigma(1) = 6$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 7$, $\sigma(5) = 5$, $\sigma(6) = 2$, and $\sigma(7) = 4$.

1. Write σ as a product of disjoint cycles.

2. Is it true that σ lies in the alternating group A_7 ?

3. What is the order of σ ?
4. Compute σ^{2018}.
5. Let $H = \langle \sigma \rangle$ be the subgroup of S_7 generated by σ. Find all generators of H.

Exercise 5. Let $n \geq 2$ be a fixed integer. Recall that the general linear group $\mathrm{GL}_n(\mathbb{R})$ is the set of $n \times n$ invertible matrices with real coefficients. Recall that it is a indeed a group when endowed with the matrix multiplication. Consider the subset $H \subseteq \mathrm{GL}_n(\mathbb{R})$ of upper triangular invertible matrices.

1. Check that H is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
2. Show that H is not a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ *(your proof should work for every n).*
3. Let $K \subseteq H$ be the set that consists of upper triangular matrices with 1 on the diagonal.

   Show that there exists an abelian group E and a group homomorphism $f : H \longrightarrow E$ (to be defined) such that $K = \mathrm{Ker}(f)$. **Deduce** that K is a normal subgroup of H and that the quotient group H/K is abelian.

Exercise 6. The ring of Gaussian integers in the following subset of $\mathbb{C}$:

$$\mathbb{Z}[i] := \{a + ib; a, b \in \mathbb{Z}\}.$$

1. Check that $\mathbb{Z}[i]$ is indeed a subring of the ring $\mathbb{C}$.
2. We consider, in the ring $\mathbb{Z}[i]$, the ideal I generated by the element $3 + i$. Determine whether the complex numbers $9 - 7i$ et $8 + i$ belong or not to I.
3. Let $a, b \in \mathbb{Z}$. Show that $a + ib \in I$, if and only if, $a + 7b$ is a multiple of 10.
4. Consider the following map:

$$f \quad : \quad \begin{array}{ccc} \mathbb{Z}[i] & \to & \mathbb{Z}/10\mathbb{Z} \\ a + ib & \mapsto & \overline{a + 7b} \end{array},$$

   where $\overline{a + 7b}$ denotes the class of the integer $a + 7b$ in $\mathbb{Z}/10\mathbb{Z}$. Show that f is a surjective ring homomorphism.
5. Show then the following ring homomorphism

$$\mathbb{Z}[i]/I \simeq \mathbb{Z}/10\mathbb{Z}.$$

6. Is I a prime ideal?