COMPREHENSIVE EXAM
IN
ANALYSIS

The Department Of Mathematics
American University Of Beirut
December 3, 2014

Instructions: Do problems **1**,**2**, and any **three** of the remaning **seven** problems.To receive credit on a problem, you must show your work and justify your conclusions.

1. If f is a real-valued function defined on $\mathbb{R}$,and $\lim_{t\to x} f(t) = L$, and $L \neq 0$, prove that $\lim_{t\to x} \frac{1}{f(t)} = \frac{1}{L}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 \sin(\frac{1}{x^2}), x \neq 0, f(0) = 0.$$

Prove that f is differentiable on $\mathbb{R}$, that its derivative is bounded but not uniformly continuous on $(0, \infty)$, and that f is uniformly continuous on $\mathbb{R}$.

3. For $x > 0$, let $F(x) = \int_0^\infty \frac{1-e^{-xt^2}}{t^2} dt$.
(a) Show that the improper integral defining F is convergent.
(b) Show that the function F is differentiable, and find an explicit formula for $F'(x)$ in terms of elementary functions.
(c) Use the result of part (b) to find an explicit expression for $F(x)$ in terms of elementary functions.

4. (a) Show that $n^n e^{-n} \leq n! \leq n^n$ for all positive integers n.
(b) Let $\{b_n\}$, $n = 1, 2, ...,$ be a sequence of positive real numbers. Suppose there is a real number β and a constant $C > 0$ so that for all $n \geq 1$

$$\frac{b_{n+1}}{b_n} = 1 + \frac{\beta}{n} + R(n) \quad \text{where} \qquad |R(n)| \leq \frac{C}{n^2}.$$

Show that, depending on β, the sequence $\{b_n\}$ has a limit which is either zero, positive, or infinite.
(c) Using the results of part (b), show that the sequence $\{\frac{n!}{n^n e^{-n}\sqrt{n}}\}, n = 1, 2, ...,$ has a finite non-zero limit.

5. If $a_n > 0$, and $\lim_{n\to\infty} a_n \sum_{k=1}^n a_k = 2$. Prove that $\lim_{n\to\infty} \sqrt{n} a_n = 1$.You may want first to find $\lim_{n\to\infty} a_n$.

6. Evaluate the integral $\int_0^\infty \frac{\log(1+x^2)}{x^{1+\beta}} dx$, where $0 < \beta < 2$. (Hint: First integrate by parts, then residues).

7. Suppose $u : \mathbb{C} \to \mathbb{R}$ is a harmonic function on the entire complex plane $\mathbb{C}$. If $u(x, y) = f(x)g(y)$ for all (x, y) where f, g are twice differentiable functions of one real variable find u. You may want to start by giving an example of one such harmonic function.

8. Discuss the pointwise and uniform convergence of the sequence of functions $\{\frac{n^2 x}{1+n^3x^2}\}$ on the following intervals: (a) $[-1, 1]$, (b) $[1, 2]$, (c) $[a, \infty), a > 0$.

9. Let f_n be a sequence of continuous real valued functions defined on $[a, b]$. If the sequence f_n converges to a function f on $[a, b]$, which of the following statements is true and which is not. In your answer, supply a proof or a counter example as the case may require.

(a) f is a continuous function on $[a, b]$.

(b) $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.