American University of Beirut
Analysis Comprehensive Exam
Fall 2015

Part I: Real Analysis

<u>Instructions:</u> Solve problems 1, 2 and any two of the problems 3, 4, 5.

Problem 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions differentiable at x_0 with $g(x_0) \neq 0$. Prove that $\dfrac{f}{g}$ is differentiable at x_0 and find an expression for $\left(\dfrac{f}{g}\right)'(x_0)$.

Problem 2. Let (X, d) be a metric space. Prove that the following statements are equivalent:

i. any Cauchy sequence in X converges.
ii. any sequence $\{x_n\}$ such that the series $\sum d(x_n, x_{n+1})$ converges is convergent.
iii. any sequence $\{x_n\}$ such that $d(x_n, x_{n+1}) < 1/n^2$, $n > 1$, is convergent.

Problem 3. Let $I := [0, 1]$ and let $f : I \to I$ be an increasing function. Consider the set

$$A := \{x \in I, f(x) \leq x\}.$$

(a) Prove that $\inf A$ exists and that $0 \leq \inf A$. Denote $\alpha := \inf A$.
(b) Prove that $f(\alpha) = \alpha$ (<u>hint:</u> notice that if $x \in A$ then $f(x) \in A$).

Problem 4.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist $a, b \geq 0$ such that $|f(x)| \leq a|x| + b$ for all $x \in \mathbb{R}$.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that that there exist $a, b \geq 0$ such that $|f(x)| \leq a|x| + b$ for all $x \in \mathbb{R}$. Prove or disprove (using an explicit counterexample) that f is uniformly continuous.

Problem 5. Let $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) := \begin{cases} n^2 x(1 - nx) \text{ if } x \in [0, \frac{1}{n}] \\ 0 \text{ otherwise.} \end{cases}$$

(a) Study the pointwise convergence of f_n on $[0, 1]$.
(b) Compute $\int_0^1 f_n(t)dt$.
(c) Study the uniform convergence of f_n on $[0, 1]$.
(d) Let $0 < a < 1$. Study the uniform convergence of f_n on $[a, 1]$.

Part II: Complex Analysis

Notations: Denote by $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\partial\Delta = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary.

Instructions: Solve the problem 6 and one of the problems 7, 8.

Problem 6.

(a) Let $R(X,Y) = P(X,Y)/Q(X,Y)$ be a rational function with no poles on $\partial\Delta$.

i. Prove that

$$\int_0^{2\pi} R(\cos t, \sin t)dt = 2\pi i \sum_{z_j \in \Delta} Res\left[\frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right), z_j\right],$$

where $Res(f, z_j)$ is the residue of f at the pole z_j.

(b) Using this method, compute the integral $\displaystyle\int_0^{2\pi} \frac{1}{a + \sin t} dt$ for $a > 1$.

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, i.e. f is holomorphic on $\mathbb{C}$.

(a) Prove that if $\iint_{\mathbb{R}^2} |f(x+iy)|^2 dxdy < \infty$ then $f(z) = 0$ for all $z \in \mathbb{C}$.

(b) Suppose that f is nonconstant. Prove that for all $\omega \in \mathbb{C}$ and for all $\varepsilon > 0$ there exists $z \in \mathbb{C}$ such that $|f(z) - w| < \varepsilon$ (do not just quote Picard theorem).

Problem 8. Let $f : \Delta \to \mathbb{C}$ be a holomorphic function on the unit disc.

(a) Suppose that $|f(z)| \leq 1$ for all $z \in \Delta$. Prove that for any integer $n \geq 1$, $|f^{(n)}(z)| \leq n!(1 - |z|)^{-n}$.

(b) Suppose that $|f(z)| < 1$ for all $z \in \Delta$. Let $0 \neq a_j \in \Delta$ such that $f(a_j) = 0$ for $j = 1, \cdots, n$. Prove that $|f(0)| \leq \Pi_{j=1}^n |a_j|$. What can you conclude if the equality holds?