American University of Beirut
Analysis Comprehensive Exam
August 2016, Duration: 3h

Part I: Real Analysis

Exercise 1. Prove rigorously that the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $K \subset \mathbb{R}$ be a compact subset. Prove that $f(K)$ is compact.

Exercise 3. Let $\sum a_n$ be a series such that $a_n > 0$ for all n. Prove that if $\limsup \frac{a_{n+1}}{a_n} < 1$ then the series $\sum a_n$ converges.

Problem 1. Let $\{x_n\}$ be a numerical sequence.

(a) Suppose that $\{x_n\}$ converges. Prove rigorously that the sequence $\{x_{n+1} - x_n\}$ converges.

(b) Suppose that $\{x_{n+1} - x_n\}$ converges to $l \in \mathbb{R}$.

i. Prove that $\left\{\frac{x_n}{n}\right\}$ converges and find its limit.

ii. Study the convergence of $\{x_n\}$ in case $l \neq 0$.

iii. Study the convergence of $\{x_n\}$ in case $l = 0$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ such that there exists $\alpha \in (0, 1/2)$ such that for any $x, y \in \mathbb{R}$

$$|f(x) - f(y)| < \alpha \left(|f(x) - x| + |f(y) - y|\right).$$

Prove that f admits an unique fixed point, i.e. a point $a \in \mathbb{R}$ satisfying $f(a) = a$.

Problem 3. Let $f : (0, \infty) \to \mathbb{R}$ be a twice differentiable function. (note: Parts (a), (b) and (c) are independent of each other)

(a) Assume that $\lim_{x\to\infty} f'(x) = \infty$. Prove that $\lim_{x\to\infty} f(x) = \infty$

(b) Assume that $\lim_{x\to\infty} f(x) = l_0 \in \mathbb{R}$ and $\lim_{x\to\infty} f''(x) = l_2$. Prove that $l_2 = 0$.

(c) Find a function f such that $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$ and f' does not admit any limit at ∞.

Problem 4. Consider the sequence of functions $f_n(x) = x(1 + n^\alpha e^{-nx})$, $\alpha \in \mathbb{R}$, defined on $[0, \infty)$.

(a) Study, according to the values of α, the pointwise converge of f_n.

(b) Study, according to the values of α, the uniform converge of f_n.

(c) Compute $\lim_{n\to\infty} \int_0^1 x(1 + \sqrt{n}e^{-nx})dx$.

Problem 5. Let $f_n : [0, \infty) \to \mathbb{R}$ be a sequence of functions converging uniformly to f.

(a) Prove or disprove that $(f_n)^2$ converges pointwise to f^2.

(b) Prove or disprove that $(f_n)^2$ converges uniformly to f^2.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove or disprove that $g \circ f_n$ converges uniformly to $g \circ f$.

Part II: Complex Analysis

Notations: Denote by $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\partial\Delta = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary.

Exercise 4. Let $f(z) = \frac{1}{z(z-1)(z+3)}$. Give the Laurent series expansion of f in each of the following annuli

(a) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$

(b) $\{z \in \mathbb{C} \mid 1 < |z| < 3\}$

Exercise 5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function on the unit disc Δ. Assume that $|f(z)| \leq 2$ for all $z \in \Delta$. Can we have $a_3 = 3$?

Problem 6.

(a) Prove that the improper integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1} dx$ converges.

(b) Using only complex analysis techniques, prove that $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$.

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Assume that there is a disc $\Delta_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ such that $f(\mathbb{C}) \cap D_r(z_0) = \emptyset$. Prove that f is constant.

(b) Assume that $|f(z)| \to \infty$ as $|z| \to \infty$. Prove that f has a finite number of zeros.

Problem 8. Let f be holomorphic function. We define the n^{th} iterate of f by $f^{\circ n} = f \circ \cdots \circ f$ (n times).

(a) Find $f^{\circ n}$ for the following functions $f(z) = z + 1$, $f(z) = \lambda z$, $f(z) = z^N$.

(b) For $c \in \mathbb{C}$, we denote by f_c the polynomial $f_c(z) = z^2 + c$. We define the *filled Julia set of* f_c, denoted by K_c, by being the set of $z \in \mathbb{C}$ such that $|f_c^{\circ n}(z)|$ does not converge to ∞ as $n \to \infty$. By definition the boundary ∂K_c is the *Julia set of* f_c

  i. Determine K_0.

  ii. Prove that $K_c = \{z \in \mathbb{C} \mid \forall n \geq 0, |f^{\circ n}(z)| \leq 1 + |c|\}$.

  iii. Deduce that K_c is compact.