American University of Beirut
Analysis Comprehensive Exam
Fall 2018
Time allowed: 3h00

Part I: Real Analysis

We denote by $\mathbb{R}^+$ the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ of positive real numbers.

Exercise 1. Let $\mathcal{R} \subset \mathbb{R}^2$ be the region in the first quadrant bounded by the curves $y = x - 3$ and $y = x^2$ and $y = 4$. Consider the solid $\mathcal{S}$ of revolution obtained by revolving the region $\mathcal{R}$ about the axis $y = 4$. Find its volume.

Exercise 2. Let $A = \big\{x \in \mathbb{R} | \ x = (-\frac{1}{2})^m - \frac{3}{n}$ for some $m, n \in \mathbb{N} \setminus \{0\}\big\}$.

(a) Prove that $\inf A$ and $\sup A$ both exist.

(b) Compute $\inf A$ and $\sup A$.

Exercise 3.

(a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b) and such that $f(a) = f(b)$. Prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.

(b) Deduce the Mean Value Theorem.

Exercise 4. Let $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $f(x) \in \mathbb{R} \setminus \mathbb{Q}$. Show, using the definition of Riemann integral, that f is not Riemann integrable.

Exercise 5. Let $f : [0, 1] \to \mathbb{R}$ be a continous function satisfying $f(0) = f(1)$.

(a) Let $n > 0$ be an integer. Prove that there exists $x_n \in [0, 1 - 1/n]$ such that $f(x_n) = f(x_n + 1/n)$

(b) Prove that the conclusion of the previous question fails if $1/n$ is replace by $\alpha \in (0, 1)$ with $1/\alpha \notin \mathbb{N}$. You may consider the function $x \to \cos\left(\frac{2\pi x}{\alpha}\right) - x(\cos\left(\frac{2\pi}{\alpha}\right) - 1)$.

Problem 1. Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^∞. Define a sequence of real functions $\{f_n\}_{n \geq 0}$ as $f_0(x) = g(x)$, $f_n(x) = \sin(f_{n-1}(x))$ for $n \geq 1$.

(a) For a certain $x_0 \in \mathbb{R}$, suppose that $f_1(x_0) > 0$. Show that $f_n(x_0) > 0$ for all $n > 1$.

(b) Show that $\{f_n\}$ converges pointwise on $\mathbb{R}$, and find the limit function.

(c) Does $\{f_n\}$ converge uniformly?

(d) Show that $\{f'_n\}$ converges pointwise.

Problem 2. Let $a_n(t)$ be a sequence of positive continuous functions $a_n : [0, 1] \to \mathbb{R}^+$ such that $\sum_n a_n(t)$ is convergent for all $t < 1$.

(a) Suppose that there exists $C > 0$ such that $\sum_n a_n(t) \leq C$ for all $t < 1$. Show that $\sum a_n(1)$ is convergent.

(b) Without the assumption in (a), show by an example that $\sum a_n(1)$ might be divergent.

(c) Suppose that each a_n is monotone increasing and $\sum_n a_n(t) \to +\infty$ as $t \to 1$. Show that $\sum a_n(1)$ is divergent.

(d) Show that the conclusion in (c) is not valid without the monotonicity assumption.

Problem 3. Let $I \subset \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$ be a function of class C^∞ and denote by $f^{(n)}$ its n-th derivative. We say that f admits an ∞-*derivative* if $\{f^{(n)}\}$ converges uniformly on I to a function $f^{(\infty)}$.

(a) Do the following functions admit or not an ∞-derivative?
  i. $f(x) = e^x$ on $(0, 1)$.
  ii. $f(x) = \sin(x)$ on $(0, 2\pi)$.
  iii. $f(x) = \log x$ on $(1, 2)$.

(b) Suppose that $f : I \to \mathbb{R}$ admits an ∞-derivative $f^{(\infty)}$. Show that $f^{(\infty)}$ is of class C^∞.

(c) Suppose that $f : I \to \mathbb{R}$ admits an ∞-derivative $f^{(\infty)}$. Show that $f^{(\infty)}(x) = Ce^x$ for some $C \in \mathbb{R}$.

Problem 4.

(a) For $\alpha, \beta \geq 1$ consider the improper integral

$$\int_e^{+\infty} \frac{1}{x^\alpha (\log x)^\beta} dx$$

Show that it is is divergent for $\alpha = \beta = 1$ and convergent otherwise.

(b) Show that the improper integral

$$\int_{e^2}^{+\infty} \frac{1}{x(\log x)(\log \log x)^\gamma} dx$$

is divergent for $\gamma = 1$ and convergent for $\gamma > 1$.

Part II: Complex Analysis

We denote by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary and by $\mathbb{H} = \{z \in \mathbb{C} \mid \Im m z > 0\}$ the upper half plane.

Exercise 6. Let γ be the boundary of the square $[0, 1] \times [0, i]$ with counterclockwise orientation. Evaluate the integrals

(a) $\int_\gamma \Re e z dz$.

(b) $\int_\gamma \Im m z dz$.

Exercise 7.

(a) Find the zeros of $\sin\left(\frac{1+z}{1-z}\right)$ in $\mathbb{D}$.

(b) Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function satisfying $f(z_n) = 0$ for a sequence $\{z_n\}$ converging in $\overline{\mathbb{D}}$. Prove or disprove that f is identically equal to 0.

Exercise 8.

(a) Show that the function $\varphi : \mathbb{H} \to \mathbb{D}$ defined by $\varphi(z) = \dfrac{z - i}{z + i}$ is a holomorphic bijection with holomorphic inverse (do not forget to show that $\varphi(\mathbb{H}) \subset \mathbb{D}$).

(b) Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ be a holomorphic function satisfying $\Im m f(z) > 0$ for all $z \in \mathbb{D} \setminus \{0\}$. Study the nature of its singularity at 0 (that is, removable, pole or essential).

Problem 5.

Let Ω be a connected domain in $\mathbb{C}$ such that $\overline{\mathbb{D}} \subset \Omega$.

(a) Let $f \in \mathcal{O}(\Omega)$ such that $f(0) = 1$ and $|f(z)| > 1$ for all $z \in \partial\mathbb{D}$. Prove that f admits a zero in $\mathbb{D}$.

(b) Let $f \in \mathcal{O}(\Omega)$ be a non constant map such that $|f(z) - 2| < 1$ for all $z \in \partial\mathbb{D}$.
  i. Show that $|f(z)| < 3$ for all $z \in \mathbb{D}$.
  ii. Let $w_0 \in \mathbb{D}$. Show that the equation $f(z) = w_0$ has no solution in $\mathbb{D}$.

(c) Let $f \in \mathcal{O}(\Omega)$ be a non constant map such that $|f(z)| = 1$ for all $z \in \partial\mathbb{D}$. Prove that f admits a zero in $\mathbb{D}$.