American University of Beirut
Analysis Comprehensive Exam
Fall 2019-2020
Time allowed: 2 hours

Part I: Real Analysis

Exercise 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0 and $f(x_0) \neq 0$. Show that $g = 1/f$ is differentiable at x_0, and find (with proof) the formula for $g'(x_0)$.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function.

(a) Show that if f is continuous at $x_0 \in \mathbb{R}$ then for every real sequence $\{x_n\}$, $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$ as $n \to \infty$.

(b) Suppose that f is not continuous at x_0. Show that there is a real sequence $\{x_n\}$ such that $x_n \to x_0$ as $n \to \infty$ but $f(x_n)$ does not converge to $f(x_0)$.

Exercise 3. Let $\{I_n\}$ be a sequence of intervals of $\mathbb{R}$ with the property that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.

(a) Suppose that each I_n is closed and bounded. Show that there exists a point $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

(b) Is the same conclusion as in (a) valid if each I_n is an open interval? What about if each I_n is an unbounded interval of the form $[a, +\infty)$.

Problem 1. Define $f_n : [1, +\infty) \to \mathbb{R}$ as $f_n(x) = \left(\frac{e}{n}\right)^n \frac{(\log(x))^n}{x}$.

(a) Show that the sequence $\{f_n\}_{n\geq 1}$ converges pointwise to a function $f : [1, +\infty) \to \mathbb{R}$, and find f.

(b) Show that f_n admits a unique point of maximum on $[1, +\infty)$.

(c) Does $\{f_n\}$ converge uniformly (i) on $[1, 1000]$? (ii) on $[1, +\infty)$?

Problem 2. Denote by X the set of *non negative* C^∞ functions $f : \mathbb{R} \to \mathbb{R}$.

(a) Suppose that f and f' are both in X. Show that either f is strictly positive, identically zero or there exists $a \in \mathbb{R}$ such that $f(x) = 0 \Leftrightarrow x \leq a$.

(b) Let f be as in (a). Show that $\lim_{x\to-\infty} f(x)$ exists.

(c) (unrelated to (a), (b)) Suppose that $f, \sqrt{f} \in X$ and $f(0) = 0$. Find $f'(0)$ and $f''(0)$.

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Given $x \in [0, 1]$, we define

$$G(x) = \sup_{t\in[0,1-x]} \int_t^{t+x} f(s)ds;$$

in other words $G(x)$ is the supremum of the integrals of f taken over intervals of length x.

(a) Show that the supremum is actually a maximum, that is, given $x \in [0, 1]$ there exists $t \in [0, 1]$ such that $G(x) = \int_t^{t+x} f(s)ds$.

(b) Suppose that f is decreasing. Show that $G(x) = \int_0^x f(s)ds$.

Part II: Complex Analysis

Exercise 4. (a) Let $\gamma = [1,0] \cup [0,i]$ be the path in $\mathbb{C}$ joining the points 1 and i through the oriented segments $[1,0]$ and $[0,i]$. Compute (directly via a convenient parametrization of γ) the integral $\int_\gamma e^z dz$.

(b) Use (a) to compute $\int_0^{\pi/2} e^{\cos(\theta)} \sin(\theta + \sin(\theta)) d\theta$. (Consider the curve $\lambda \subset \mathbb{C}$ parametrized as $\lambda(\theta) = e^{i\theta}$, $0 \leq \theta \leq \pi/2$).

In the following, we employ the standard O notation: $p(x) = O(q(x))$ if there exists a constant $C > 0$ such that $|p(x)| \leq C|q(x)|$.

Exercise 5. Let f, g be two holomorphic functions defined on a bounded neighborhood of 0. Suppose that the Taylor expansions of f, g can be written around 0 as follows: $f(z) = z + az^k + P(z)$, $\quad g(z) = z + bz^k + Q(z)$ for certain $k \geq 2$, $a, b \in \mathbb{C}$ and $P(z), Q(z) = O(z^{k+1})$. Show that $f \circ g(z) = z + cz^k + R(z)$ with $R(z) = O(z^{k+1})$, and find c.

Given a function ψ, we denote by $\psi^{\circ j}$ the composition of ψ with itself performed j times: $\psi^{\circ 2} = \psi \circ \psi$, $\psi^{\circ (j+1)} = \psi \circ \psi^{\circ j}$ for $j \geq 2$.

Problem 4. Let $D \subset \mathbb{C}$ be an open set, $0 \in D$, and let $\phi : D \to \mathbb{C}$ be holomorphic. We assume that $j \in \mathbb{N}$ is chosen in such a way that $\phi^{\circ j}$ is defined on D.

(a) Suppose that $\phi(z) = z + az^k + O(z^{k+1})$ for certain $k \geq 2$, $a \in \mathbb{C}$. Find $\frac{d^k \phi^{\circ j}}{dz^k}(0)$. (You can use Ex. 5).

(b) Let $\Delta_r \subset D$ be a disc of radius r and center 0, and let $M_j = \sup_{\Delta_r} |\phi^{\circ j}|$. Show that $M_j \geq j r^k |a|$. (*Hint*: take the k-th derivative with respect to z of the Cauchy formula $\phi^{\circ j}(z) = \frac{1}{2\pi i} \int_{\Delta_r} \frac{\phi^{\circ j}(\zeta)}{\zeta - z} d\zeta$).

(c) Assume now that D is bounded and $\phi : D \to D$ is bijective. Use (b) to prove the following: if $\phi(0) = 0$ and $\phi'(0) = 1$ then $\phi(z) = z$. (*Hint*: if $\phi(z) \neq z$, consider the first non-zero term in its Taylor expansion after z. What can be said about M_j?).