American University of Beirut
Analysis Comprehensive Exam
Fall 2020-2021
Time allowed: 2h30

Part I: Real Analysis

Exercise 1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^x f(t)dt = \int_x^1 f(t)dt$$

for all $x \in [0, 1]$. Show that f is identically zero.

Exercise 2. Let $\{a_n\}$ be a non-negative sequence. Assume that $\limsup_{n\to\infty} n^2 a_n$ is finite. Show that $\sum_{n=1}^{\infty} a_n$ converges.

Remark: If you are not familiar with limsup, show that if the sequence $n^2 a_n$ converges then $\sum a_n$ converges for partial credits.

Exercise 3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

(1) Suppose that for all $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a, b]$ such that $f(c) = 0$.

(2) Is the same result true if we replace $[a, b]$ with (a, b)? Prove it if it is true, or give a counterexample if false.

Problem 1.

(1) Suppose that $\{f_n\}$ is a sequence of bounded real functions converging uniformly on a set $A \subset \mathbb{R}$ to $f : A \to \mathbb{R}$. Show that f is bounded.

(2) Let $g_n(x) = \dfrac{x^n}{1 + x^n}$. Show that the series $\sum_{n=0}^{\infty} g_n$ converges pointwise on $[0, 1)$.

(3) For all $0 < a < 1$, show that $\sum_{n=0}^{\infty} g_n$ converges uniformly on $[0, a]$.

(4) Let $g(x) = \sum_{n=0}^{\infty} g_n(x)$. Show that $g(x) \to +\infty$ as $x \to 1^-$. (*Hint*: compare to a geometric series).

(5) Deduce whether the series $\sum_{n=0}^{\infty} g_n$ converges uniformly on $[0, 1)$.

(6)* Assume that a sequence of continuous functions $h_n : [a, b] \to \mathbb{R}$ converges uniformly *on* (a, b) to a function h. Does it follow that h extends continuously to $[a, b]$?

Problem 2. Let $\{a_n\}, \{b_n\}$ be two real sequences and define

$$\sigma_n = \frac{1}{n+1} \sum_{j=0}^{n} a_j b_{n-j}.$$

(1) Suppose b_n is bounded and $\lim_{n\to\infty} a_n = 0$. Show that $\lim_{n\to\infty} \sigma_n = 0$.

(2) Give an example of sequences a_n and b_n such that $\lim_{n\to\infty} a_n = 0$, b_n is unbounded, and σ_n does not converge to 0.

(3) Suppose $\lim_{n\to\infty} a_n = \alpha \in \mathbb{R}$, $\lim_{n\to\infty} b_n = \beta \in \mathbb{R}$. Show that $\lim_{n\to\infty} \sigma_n = \alpha\beta$. (*Hint*: define $a'_j = a_j - \alpha$ and replace a_j by $a'_j + \alpha$).

Problem 3.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, and let $m = f'(0)$. Show that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$(m-\epsilon)x < f(x) < (m+\epsilon)x \text{ for } 0 < x < \delta \text{ and}$$
$$(m+\epsilon)x < f(x) < (m-\epsilon)x \text{ for } -\delta < x < 0$$

(that is, the graph of f locally lies between the lines of slope $m-\epsilon$ and $m+\epsilon$).

Given two real functions f, g, their *join* $f \wedge g$ is defined as $(f \wedge g)(x) = \max\{f(x), g(x)\}$. In what follows, $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions with $f(0) = g(0) = 0$.

(2) Suppose $f'(0) > g'(0)$. Show that there exists $\eta > 0$ such that

$$(f \wedge g)(x) = f(x) \text{ for } 0 < x < \eta, \quad (f \wedge g)(x) = g(x) \text{ for } -\eta < x < 0.$$

(3) Suppose $f \wedge g$ is differentiable at 0. Show that $f'(0) = g'(0)$.

(4) Suppose $f'(0) = g'(0)$. Show that $f \wedge g$ is differentiable at 0.

(5) Find two *non-differentiable* functions $h, k : \mathbb{R} \to \mathbb{R}$ such that $h \wedge k$ is differentiable.

Part II: Complex Analysis

Exercise 4. We will denote by $\Delta_r \subset \mathbb{C}$ the (open) disc of center 0 and radius $r > 0$, and by $b\Delta_r$ its boundary.

(1) Assume that f is holomorphic on a neighborhood of the closed annulus $\overline{\Delta}_2 \setminus \Delta_1$, and suppose that $|f(z)| \leq 2$ for $z \in b\Delta_2$, $|f(z)| = 1$ for $z \in b\Delta_1$. Show that $|f(z)| \leq |z|$ for $z \in \overline{\Delta}_2 \setminus \Delta_1$. (*Hint*: consider the function $g(z) = f(z)/z$).

(2) Suppose that the function f from part (1) is actually holomorphic on a neighborhood of $\overline{\Delta}_2$, and $f(0) = 0$. Show that $f(z) = e^{i\theta} z$ for a fixed $\theta \in \mathbb{R}$.

Problem 4.

(1) Show that $|e^w| = e^{\operatorname{Re} w}$ for all $w \in \mathbb{C}$.

(2) Consider the meromorphic function $f(z) = \frac{e^{iz}}{z^2+1}$, and for $R > 0$ let C_R be the *upper half* of the circle of center 0 and radius $R > 1$. Show that $|f(z)| \leq 1/(R^2 - 1)$ for all $z \in C_R$.

(3) Apply the residue theorem to the function f in part (1) to compute

$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2+1} dx.$$