American University of Beirut, Mathematics Comprehensive Exam, Spring 2012

Problem 1. Let $\{x_n\}$ be a convergent sequence of real numbers, with $\lim x_n = a$.
Show that $\lim \left(1 + \frac{x_n}{n}\right)^n = e^a$.

Problem 2. Let $\{f_n(x) : \mathbf{R} \to \mathbf{R}\}$ be a sequence of continuous functions converging uniformly to a function $g(x)$. Also suppose that $\{x_n\}$ is a sequence converging to a. Show that $\lim_n f_n(x_n) = g(a)$.

Problem 3. For an integer $m \geq 0$, define $I_m = \int_0^{\pi/2} \sin^m x\, dx$.
a) Compute I_0, I_1, and I_2.
b) Use integration by parts to find a relation between I_m and I_{m-2}.
c) Find the value of I_m for all m.

Problem 4. For $a \geq 0$, define

$$f(a) = \int_{x=1}^{\infty} \frac{1}{x^4 + ax} dx.$$

a) Show that f is a continuous function of a.
b) Show that $\lim_{a\to\infty} f(a) = 0$.

Problem 5. Use complex analysis to compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx.$$

Problem 6. For what values of $z \in \mathbf{C}$ does the following series converge?

$$\sum_{n=0}^{\infty} \frac{2^n + n^2}{3^n + n^3} z^n$$

Problem 7. Compute

$$\lim_{t\to 0} \frac{e^{t^3} - 1 - t^3}{\sin(t^2) - t^2}.$$

Problem 8. Give a reasonable interval $I \subset \mathbf{R}$ so that the approximation

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

is valid for $x \in I$ with an error whose absolute value is less than 10^{-4}.

Problem 9. Let $S = \{(x, y) \in \mathbf{R}^2 \mid x, y \geq 0, 1 \leq x + y \leq 2\}$ (see figure 1 on page 2). This is the region in the first quadrant between the lines $x + y = 1$ and $x + y = 2$.
a) Compute $\iint_S x\, dA$.
b) Set up but do not evaluate the above integral in polar coordinates (r, θ).

Problem 10. Let C be the closed curve in the plane that goes from $(0, 0)$ to $(2, 4)$ along the parabola $y = x^2$, and then returns from $(2, 4)$ to $(0, 0)$ along the line $y = 2x$. (See figure 2 on page 2.)
a) Compute $\int_C y\, dx + x^2\, dy$ directly as a line integral.
b) Do the computation again using Green's theorem.

Problem 11. Suppose $X \subset \mathbf{C}$ is a star-shaped region in the complex plane (see figure 3 on page 2). Let $f, g : X \to \mathbf{C}$ be holomorphic functions such that

$$f(z)^2 + g(z)^2 = 1, \quad \forall z \in X.$$

Show that there exists a holomorphic function $\varphi : X \to \mathbf{C}$ such that $f(z) = \cos\varphi(z)$ and $g(z) = \sin\varphi(z)$.

Figure 1:

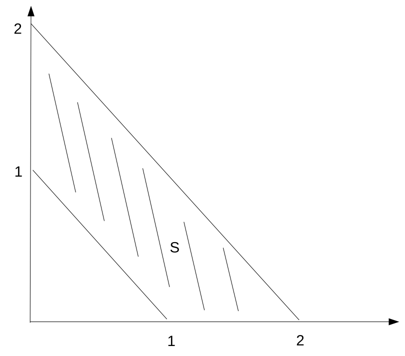

Figure 2:

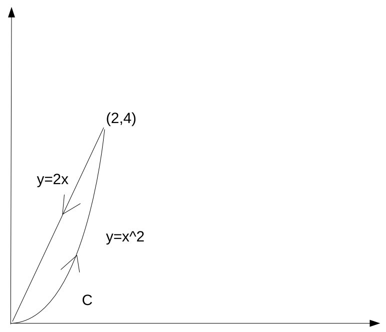

Figure 3: