American University of Beirut
Analysis Comprehensive Exam
April 2015, Duration: 90 min.

Part I: Real Analysis

Instructions: Solve any two of the problems 1, 2, 3 and any two of the problems 4,5,6,7.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$. Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbb{R}$.

Problem 2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions differentiable at x_0. Prove that fg is differentiable at x_0 and that $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Problem 3. Compute the volume of the solid region in $\mathbb{R}^3$ bounded by the surface S defined by $z = 2\sqrt{x^2 + y^2}$, $x \geq 0$, $y \geq 0$, $0 \leq z \leq 1$.

Problem 4. Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We define a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ by

$$\sigma_n := \frac{1}{n+1}\sum_{k=0}^{n} a_k.$$

(a) Assume that $\{a_k\}_{k\in\mathbb{N}}$ converges to a limit ℓ. Prove that $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to ℓ (hint: start by considering the case $\ell = 0$).
(b) Assume that $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to ℓ. Prove or disprove (via a counterexample) that $\{a_k\}_{k\in\mathbb{N}}$ must converge to ℓ.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
(a) Supose that f is twice differentiable at x_0. Prove that

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

(b) Assume that the limit of $\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$ as $h \to 0$ exists. Prove or disprove (via a counterexample) that f is twice differentiable at x_0.

Problem 6. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions defined by $f_n(x) = \frac{nx^2}{1+nx}$ if $x \geq 0$ and by $f_n(x) = \frac{nx^3}{1+nx^2}$ if $x < 0$.
(a) Prove that f_n is of class $\mathcal{C}^1$ for all $n \in \mathbb{N}$.
(b) Study the pointwise and uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$.
(c) Study the pointwise and uniform convergence of $\{f_n'\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$.

Problem 7.
(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^2 - y^2, 2xy)$. By direct parametrization, compute the following line integral

$$\int_{\mathbb{S}^1} (x^2 - y^2)\mathrm{d}x - 2xy\mathrm{d}y,$$

where $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ is the unit circle in $\mathbb{R}^2$.

(b) Let $f = (u, v) : D \to \mathbb{R}^2$ be a function of class $\mathcal{C}^1$ defined on an open set $D \subset \mathbb{R}^2$. Assume that f satisfies $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Compute the following line integral by using Green's theorem

$$\int_{\partial K} u\mathrm{d}x - v\mathrm{d}y,$$

where $K \subset D$ is a compact subset with smooth boundary.

Part II: Complex Analysis

<u>Instructions:</u> Solve the problem 8 and one of the problems 9,10.

Problem 8.

(a) Let $P(x)/Q(x)$ be a rational function with $d = \deg P - \deg Q \geq 2$ and with no poles on $(-\infty, +\infty)$.

i. Prove that the improper integral $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx$ exists.

ii. Prove that

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{\Im m z_j > 0} Res(P/Q, z_j),$$

where $Res(P/Q, z_j)$ is the residue of P/Q at the pole z_j.

(b) Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx$.

Problem 9.

(a) Let f be a holomorphic function on a nonempty open connected set $D \subset \mathbb{C}$. Let $z_0 \in D$ be a local minimum of $|f|$. Prove that either $f(z_0) = 0$ or f is constant on D.

(b) Let c be a positive real number. Prove or disprove that there exists a holomorphic function f on the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ such that $|f(z)|^2 = |z|^2 + c$ for all $z \in \mathbb{D}$.

Problem 10.

(a) Let $N > 0$ be an integer. Find all the entire (holomorphic on $\mathbb{C}$) functions f satisfying $|f(z)| \leq 1 + |z|^N$ for all $z \in \mathbb{C}$.

(b) Find all the entire functions f satisfying $|f(z)| = |z|^2$ for all $z \in \mathbb{C}$.