American University of Beirut
Analysis Comprehensive Exam
Spring 2017
Time allowed: 3h

Part I: Real Analysis

Exercise 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Prove that $g \circ f$ is continuous.

Exercise 2. Let $\{a_n\}$ be an increasing sequence bounded above. Prove that $\{a_n\}$ is convergent.

Exercise 3. Let $\{a_n\}$ be a positive sequence. Suppose that $\lim_{n\to\infty}(a_n \sum_{k=1}^{n} a_k) = 5$. Prove that $\{a_n\}$ converges and find its limit.

Problem 1. A function $f : I \to \mathbb{R}$, defined on an interval, is called *Lipschitz* if there exists a constant $C > 0$ such that for all $x, y \in I$ we have

$$|f(x) - f(y)| \leq C|x - y|.$$

(a) Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function. Prove that if f is Lipschitz then f is uniformly continuous.
(b) Let $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = |x|$. Prove that f_1 is Lipschitz on $\mathbb{R}$.
(c) Let $f_2 : [a, b] \to \mathbb{R}$ defined by $f_2(x) = x^2$. Prove that f_2 is Lipschitz on $[a, b]$.
(d) Let $f_3 : \mathbb{R} \to \mathbb{R}$ defined by $f_3(x) = x^2$. Prove that f_3 is not Lipschitz on $\mathbb{R}$.

Problem 2. Let $\{f_n\}$ be the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \sin(x/n)$.
(a) Show that f_n converges pointwise and find the limit function f.
(b) Does f_n converge uniformly on the set X, in the case where
  i. $X = [0, \pi]$?
  ii. $X = \mathbb{R}$?

Problem 3. Find the limit as $n \to \infty$ of
(a) $\displaystyle\int_0^1 t^n dt.$
(b) $\displaystyle\int_0^1 \frac{t^n}{1+t} dt.$
(c) $\displaystyle\int_0^1 \frac{nt^n}{1+t} dt.$

Problem 4. Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(x)/x \to 0$ as $x \to \infty$.
(a) For any $\epsilon > 0$ show that there exists $\xi \in \mathbb{R}$ such that $|f'(\xi)| < \epsilon$.
(b) Prove or disprove that $f'(x) \to 0$ as $x \to \infty$?
(c) Find the limit of $x - f(x)$ as $x \to \infty$.

Problem 5.
(a) Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions such that $f(0) = 0, f(1) = 1, g(0) = 1, g(1) = 0$. Show that there exists $x_0 \in (0, 1)$ such that $f(x_0) = g(x_0)$.
(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuous function. Show that there exists $x_0 \in \mathbb{R}$ such that $h(x_0 + \pi) = h(x_0)$.

Part II: Complex Analysis

Exercise 4. Let $D = \mathbb{C} \setminus \{0, 1\}$. Consider the function $f(z) = \frac{1}{z(z-1)}$. Compute $\displaystyle\int_{\partial T} f(z)dz$ for any triangle T with boundary ∂T contained in D.

Exercise 5. Recall that a holomorphic function f has a *zero of order* k at $z_0 \in \mathbb{C}$ if in a neighborhood of z_0 it can be written as $f(z) = \alpha(z - z_0)^k + (z - z_0)^{k+1}h(z)$, where $\alpha \neq 0$ and $h(z)$ is holomorphic on a neighborhood of z_0. Suppose that f has a zero of order k at z_0. Show that f'/f has a simple pole at z_0 with residue k.

Problem 6. For $t \in \mathbb{R}$ define $f_t : [0, \infty) \to \mathbb{R}$ by $f_t(x) = \frac{1}{x^2+t^2}$.

(a) Show that the integral $\int_0^{+\infty} f_t(x)dx$ is finite for $t \neq 0$ and it is not convergent for $t = 0$.

(b) For $t \neq 0$, use the residue theorem to compute $\int_{-\infty}^{+\infty} f_t(x)dx$ as a function of t, and use this to evaluate $\int_0^{+\infty} f_t(x)dx$.

(c) Deduce that $\int_0^{+\infty} f_t(x)dx \to +\infty$ as $t \to 0$.

Problem 7.

(a) Let $w \in \mathbb{C}$. Prove that $|e^w| = e^{\Re e w}$.

(b) Let $f = u + iv : \mathbb{C} \to \mathbb{C}$ be a entire function such that $xu(x + iy) - yv(x + iy) \leq 0$. Prove that f is constant and determine the constant.