American University of Beirut
Analysis Comprehensive Exam
Spring 2018
Time allowed: 3h00

Part I: Real Analysis

We denote by $\mathbb{R}^+$ the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ of positive real numbers.

Exercise 1. Prove using the $\varepsilon - \delta$ defintion of continuity that the function f defined on $\mathbb{R}$ by $f(x) = x^2$ is continuous on $\mathbb{R}$.

Exercise 2. Let $K \subset \mathbb{R}$ be a compact subset and let $f : K \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous.

Exercise 3.

(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Assume that $f\left(\dfrac{3}{4}\right) > 0$. Prove that there is $\delta > 0$ such that $\displaystyle\int_{\frac{3}{4}-\delta}^{\frac{3}{4}+\delta} f(x)dx > 0$.

(b) Is the previous statement still true in case f is only assumed to be Riemann integrable?

Exercise 4. Let f be a differentiable function on $[1, 3]$ satisfying $-1 \leq f'(x) \leq 2$ for all $x \in [1, 3]$ and $f(3) = 0$. Can we have $f(1) = -8$?

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that $f([a, b])$ is a compact interval.

Problem 1. Recall that $\mathbb{R}^+$ is the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ of positive real numbers. For any subset $A \subset \mathbb{R}^+$ we define

$$A' = \{|x - y| : x, y \in A, x \neq y\}.$$

Note that by definition $A' \subset \mathbb{R}^+$.

(a) If $\sup A$ is finite, show that $\sup A' \leq \sup A$.

(b) If $\sup A$ is finite, show that $\sup A' = \sup A$ if and only if $\inf A = 0$.

(c) Find a set $A \subsetneq \mathbb{R}^+$ such that $A' = A$.

(d) Suppose that $\inf A = 0, \sup A = +\infty$ and $A' = A$. Does it follow that $A = \mathbb{R}^+$?

Problem 2. Recall that when (X, d) is a metric space, a map $f : X \to X$ is a *contraction* if there exists a constant $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$.
Let S be the set of all continuous functions $f : [0, 1] \to [0, 1]$. Given $f \in S$, define a new function Tf as

$$(Tf)(x) = \int_0^{f(x)} f(t)dt.$$

(a) Show that $Tf \in S$ for all $f \in S$.

(b) Let $f_0 \in S$ and define $f_1 = Tf_0, f_2 = Tf_1, \ldots, f_{n+1} = Tf_n$. Show that the sequence $\{f_n\}$ converges uniformly to either (the constant function) 1 or 0.

(c) Find the function(s) $f \in S$ such that $Tf = f$. Is T a contraction on S endowed with the distance $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$?

Problem 3. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function, and suppose that $|f(x)| \leq \frac{1}{1+|x|}$ for all $x \in \mathbb{R}^+$.
(a) Show that the series $\sum_{n\geq 0} f(2^n x)$ converges to a function $g(x)$.
(b) Show that g is continuous.
(c) Show that $f(x) = g(x) - g(2x)$.

Problem 4. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Define

$$g(x) = \int_1^x x f(t) dt$$

and

$$h(x) = \int_1^{\frac{1}{x}} f(t) dt.$$

Prove that the derivative of g is decreasing if and only the one of h is.

Part II: Complex Analysis

We denote by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary.

Exercise 6. Evaluate the following integrals
(a) $\displaystyle\int_{\partial\mathbb{D}} \frac{\cos(z)}{z(z-2)^2} dz$
(b) $\displaystyle\int_{\partial D(0,2)} \frac{2\cos(z)}{z(z-1)} dz$ where $\partial D(0, 2)$ denotes the boundary of the disc centered at 0 and of radius 2.
(c) $\displaystyle\int_{\partial\mathbb{D}} \frac{(1 + z - e^z)\cos z}{z^2} dz$

Exercise 7. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisying $f(\frac{1}{n}) = \frac{1}{n^2}$ for all positive interger n. Determine f.

Exercise 8.
By integrating the function $e^{\frac{1}{z}}$ over the boundary of the unit disc find the values of the following two integrals:

$$\int_0^{2\pi} e^{\cos t} cos(t - \sin t) dt \quad \text{and} \quad \int_0^{2\pi} e^{\cos t} sin(t - \sin t) dt$$