American University of Beirut
Analysis Comprehensive Exam
Spring 2019
Time allowed: 2h00

Part I: Real Analysis

We denote by $\mathbb{R}^+$ the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ of positive real numbers.

Exercise 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point x_0. Using the definition, show that the product fg is also continuous at x_0.

Exercise 2. Let $\{a_n\}$ be a real sequence.

(a) State the Cauchy criterion for the convergence of a_n in $\mathbb{R}$.

(b) Show that if a_n is convergent, then it satisfies the Cauchy criterion.

Problem 1. Let $\{a_n\}$, $\{b_n\}$ be two real sequences, and define $c_n = a_n - b_n$, $d_n = a_n^2 - b_n^2$. Which of the following statements are true? Give either a proof or a counterexample.

(a) If $\lim_{n\to\infty} c_n = 0$ then $\lim_{n\to\infty} d_n = 0$.

(b) If a_n is bounded and $\lim_{n\to\infty} c_n = 0$ then $\lim_{n\to\infty} d_n = 0$.

(c) Assume a_n is increasing and b_n is decreasing. If c_n is bounded then it is convergent.

(d) If the series $\sum_{n=0}^{\infty} c_n$ converges absolutely and a_n is bounded then the series $\sum_{n=0}^{\infty} d_n$ converges absolutely.

Problem 2.

Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$.

(a) Suppose first that $f(-1) = f(0) = f(1)$. Using the mean value theorem repeatedly, show that there exists $\xi \in (-1, 1)$ such that $f''(\xi) = 0$.

(b) Without the assumption of (a), find the equation $g(x) = a_2x^2 + a_1x + a_0$ of the parabola passing through $(-1, f(-1))$, $(0, f(0))$ and $(1, f(1))$. When does it degenerate into a straight line?

(c) Prove the following "second order version of the mean value theorem" : there exists a point $\xi \in (-1, 1)$ such that $f''(\xi) = (f(1) - f(0)) - (f(0) - f(-1))$. (You may want to consider the function $h(x) = f(x) - g(x)$).

Problem 3.
Let $f : \mathbb{R} \to \mathbb{R}$ be a positive continuous function such that $\lim_{x\to+\infty} f(x) = 0$. For $n \in \mathbb{N}$, define $g_n(x) = f(x+n)$. Which of the following statements are true? Give either a proof or a counterexample.

(a) $g_n \to 0$ as $n \to \infty$, pointwise on $\mathbb{R}$.
(b) $g_n \to 0$ as $n \to \infty$, uniformly on $\mathbb{R}$.
(c) $g_n \to 0$ as $n \to \infty$, uniformly on $[-1, 1]$.
(d) $\int_{-1}^{1} g_n(t)dt \to 0$ as $n \to \infty$.
(e) $(g_n)^{1/n} \to 0$ as $n \to \infty$, pointwise on $\mathbb{R}$.

Part II: Complex Analysis

We denote by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary. For $a \in \mathbb{C}$ and $r > 0$, we set $D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$.

Exercise 3. Evaluate the following integrals

(a) $\displaystyle\int_\gamma (12z^2 - 6iz)dz$ where the path γ is composed of two segments $[0,1] \cup [1, 1+i]$.
(b) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx$ (explain first why that integral is convergent).

Exercise 4.

(a) Let $a \in \mathbb{C}$ and $r > 0$. Show that any holomorphic function $f : \mathbb{C} \to \mathbb{C} \setminus \overline{D(a,r)}$ is constant.
(b) Let f be a nonconstant entire function (i.e. holomorphic on $\mathbb{C}$). Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
(c) Find a nonconstant entire function such that $f(\mathbb{C}) \neq \mathbb{C}$.

Exercise 5. Let Ω be a connected open set in $\mathbb{C}$ such that $\overline{\mathbb{D}} \subset \Omega$. Let $f \in \mathcal{O}(\Omega)$ such that $f(0) = 1$ and $|f(z)| > 1$ for all $z \in \partial\mathbb{D}$. Prove that f admits a zero in $\mathbb{D}$.