This textbook explores both the theoretical foundation of the Finite Volume Method (FVM) and its applications in Computational Fluid Dynamics (CFD). Readers will discover a thorough explanation of the FVM numerics and algorithms used in the simulation of incompressible and compressible fluid flows, along with a detailed examination of the components needed for the development of a collocated unstructured pressure-based CFD solver. Two particular CFD codes are explored. The first is uFVM, a three-dimensional unstructured pressure-based finite volume academic CFD code, implemented within Matlab®. The second is OpenFOAM®, an open source framework used in the development of a range of CFD programs for the simulation of industrial scale flow problems.

With over 220 figures, numerous examples and more than one hundred exercises on FVM numerics, programming, and applications, this textbook is suitable for use in an introductory course on the FVM, in an advanced course on CFD algorithms, and as a reference for CFD programmers and researchers.
The Process

Physical Domain

Physical Phenomena

Domain Modeling

Physical Modeling

Set of Governing Equations Defined on a Computational Domain

Domain Discretization

Equation Discretization

System of Algebraic Equations

Solution Method

Numerical Solutions

Structured Grids
Cartesian, Non-Orthogonal
Block Structured grids
Unstructured Grids
Chimera Grids

Finite Difference
Finite Volume
Finite Element
Boundary Element

Combinations of Multigrid Methods
Iterative Solvers
Coupled-Uncoupled
\[ -\nabla \cdot \left( k \nabla T \right) = \dot{q} \]

\[ T_{\text{sink}} \quad \text{insulated} \]

\[ \frac{\partial \phi}{\partial t} + \nabla \cdot \left( \rho \phi \right) = 0 \]

\[ \sum_{i=1}^{n} a_i \phi_i = b_c \]
Balance Form

Consider the steady state conservation equation of a scalar

\[
\nabla \cdot (\rho \mathbf{v} \phi) = \nabla \cdot (\Gamma \nabla \phi) + Q^\phi
\]

\[
\begin{align*}
\iiint_V \nabla \cdot (\rho \mathbf{v} \phi) dV &= \iiint_V \nabla \cdot (\Gamma \nabla \phi) dV + \iiint_V Q dV \\
\oint_{\partial V} (\rho \mathbf{v} \phi) \cdot d\mathbf{S} &= \oint_{\partial V} (\Gamma \nabla \phi) \cdot d\mathbf{S} + \iint_V Q dV \\
\sum_{f = \text{faces}(V)} \int_f (\rho \mathbf{v} \phi) \cdot d\mathbf{S} &= \sum_{f = \text{faces}(V)} \int_f (\Gamma \nabla \phi) \cdot d\mathbf{S} + \iint_V Q dV
\end{align*}
\]
Face Flux Integration

\[
\sum_{f=\text{faces}(V)} \int (\rho v \phi) \cdot dS
\]

\[
\int J \cdot dS = \sum_{ip(f)} (J \cdot S)_{ip} = \sum_{ip(f)} (J_{ip} \cdot w_{ip} S_f)
\]

\[
(\rho u \phi)_{\text{FluxT}_f} \cdot S_f
\]

one integration point

two integration points

three integration points

\[
(\rho u \phi)_{1(f)} \cdot w_{1(f)} S_f + (\rho u \phi)_{2(f)} \cdot w_{2(f)} S_f
\]
Source Integration

\[
\text{FluxT} = \int \int_{V_c} Q \, dV = \sum_{ip(C)} Q_{c_{ip}} V_{c_{ip}} = \sum_{ip(C)} \left( Q_{c_{ip}} \omega_{ip} \right) V_c = \text{FluxC } \phi_C + \text{FluxV}
\]

\[
Q_c V_c = Q_{c_1} V_{c_1} + Q_{c_2} V_{c_2} + Q_{c_3} V_{c_3} + Q_{c_4} V_{c_4}
\]

- **one integration point**
- **four integration points**
- **nine integration points**
Flux Linearization

\[
\mathbf{J}_f^\phi \cdot \mathbf{S}_f = \text{Flux}T_f = \text{Flux}C_f \phi_C + \text{Flux}F_f \phi_F + \text{Flux}V_f
\]

\[
\sum_{f \sim \text{nb}(C)} (\mathbf{J}_f^\phi \cdot \mathbf{S}_f) = \sum_{f \sim \text{nb}(C)} (\text{Flux}T_f)
\]

\[
= \sum_{f \sim \text{nb}(C)} (\text{Flux}C_f \phi_C + \text{Flux}F_f \phi_F + \text{Flux}V_f)
\]

\[
Q^\phi V_C = \text{Flux}T
\]

\[
= \text{Flux}C \phi_C + \text{Flux}V
\]

\[
a_C \phi_C + \sum_{F \sim \text{NB}(C)} (a_F \phi_F) = b_C
\]

\[
a_C = \sum_{f \sim \text{nb}(C)} \text{Flux}C_f - \text{Flux}C
\]

\[
a_F = \text{Flux}F_f
\]

\[
b_C = -\sum_{f \sim \text{nb}(C)} \text{Flux}V_f + \text{Flux}V
\]
Boundary Conditions

Value Specified (Dirichlet Boundary Condition)

\[ \mathbf{J}^\phi_b \cdot \mathbf{S}_b = \mathbf{J}^{\phi,C}_b \cdot \mathbf{S}_b \]
\[ = (\rho \mathbf{v} \phi)_b \cdot \mathbf{S}_b \]
\[ = \text{FluxC}_b \phi_C + \text{FluxV}_b \]
\[ = (\rho_b \mathbf{v}_b \cdot \mathbf{S}_b) \phi_b = \dot{m}_f \phi_{b,\text{specified}} \]

\[ \text{FluxC}_b = 0 \]
\[ \text{FluxV}_b = \dot{m}_f \phi_{b,\text{specified}} \]

Flux Specified (Neumann Boundary Condition)

\[ \mathbf{J}^\theta_b \cdot \mathbf{S}_b = \mathbf{J}^\theta_b \cdot \mathbf{n}_b \cdot \mathbf{S}_b \]
\[ \text{Specified flux} \]
\[ = q_{b,\text{specified}} \mathbf{S}_b \]

\[ \text{FluxC}_b = 0 \]
\[ \text{FluxV}_b = q_{b,\text{specified}} \mathbf{S}_b \]
FVM Accuracy
Spatial Variation

\[ \phi(x) = \phi_p + (x - x_p) \cdot (\nabla \phi)_p + \frac{1}{2} (x - x_p)^2 : (\nabla \nabla \phi)_p \]

\[ + \frac{1}{3!} (x - x_p)^3 :: (\nabla \nabla \nabla \phi)_p + \ldots \]

\[ + \frac{1}{n!} (x - x_p)^n ::: (\nabla \nabla \nabla \nabla \phi)_p + \ldots \]

\[ \phi(x) = \phi_p + (x - x_p) \cdot (\nabla \phi)_p + O(\Delta x^2) \]
Mean Value Theorem

Cell Value flux

\[
\bar{\phi}_c = \frac{1}{V_c} \int \phi dV
\]

\[
= \frac{1}{V_c} \int [\phi_c + (x - x_c) \cdot (\nabla \phi)_c + O(|x - x_c|^2)] dV
\]

\[
= \frac{\phi_c}{V_c} \int dV + \frac{1}{V_c} \int (x - x_c) \cdot (\nabla \phi)_c dV + \frac{1}{V_c} \int O(|x - x_c|^2) dV
\]

\[
= \phi_c + O(|x - x_c|^2)
\]
Mean Value Theorem

convective flux

\[
\left( \rho \nu \cdot S_f \right) \phi_f = \int_f (\rho v \phi) \cdot dS \\
= \int_f \rho_f v_f \left[ \phi_f + (x - x_f) \cdot (\nabla \phi)_f + O \left( |x - x_f|^2 \right) \right] \cdot dS \\
= \rho_f v_f \phi_f \cdot \int_f dS + \int_f (x - x_f) \cdot (\nabla \phi)_f \rho_f v_f \cdot dS + \int_f O \left( |x - x_f|^2 \right) \rho_f v_f \cdot dS \\
= (\rho_f v_f \cdot S_f) \left[ \phi_f + O \left( |x - x_f|^2 \right) \right]
\]

diffusion flux

\[
\int_f (\Gamma^\phi \nabla \phi) \cdot dS = \int_f \left[ (\Gamma^\phi \nabla \phi)_f + (x - x_f) \cdot (\nabla (\Gamma^\phi \nabla \phi))_f + O \left( |x - x_f|^2 \right) \right] \cdot dS \\
= (\Gamma^\phi \nabla \phi)_f \cdot dS + \int_f (x - x_f) dS \cdot (\nabla (\Gamma^\phi \nabla \phi))_f + O \left( |x - x_f|^2 \right) \\
= (\Gamma^\phi \nabla \phi)_f \cdot S_f + O \left( |x - x_f|^2 \right)
\]
Properties of the Discretized Equations
Conservation

\[
\text{Flux}_{C} c f + \text{Flux}_{F} g f + \text{Flux}_{V} f
\]

\[
\text{Flux}_{T} f = - \text{Flux}_{T} f
\]
Transportiveness
Other Properties

Consistency

Stability

Economy

Boundedness of the Interpolation Profile
Variable Arrangement
Variable Arrangement

Cell-centered

Vertex-centered
Other Issues
Matrix Connectivity
ufvm Connectivity

Connectivity

1→[3]
2→[3,7]
3→[1 2 5 4]
4→[3 6]
5→[3 6 7]
6→[4 5]
7→[2 5]
OpenFoam Connectivity

\[
\begin{bmatrix}
\text{diag()} & \text{lower()} & \text{upper()}
\end{bmatrix}
\]

lduMatrix