Solving the Navier-Stokes Equations

Chapter 15
For the solution of the scalar equation we have assumed that the velocity field is known

\[ m = \rho v \cdot S \]

How do we actually compute the velocity field is the subject of this lecture
Velocity Pressure Coupling
Navier-Stokes Equations

$$\frac{\partial (\rho \phi)}{\partial t} + \nabla \cdot (\rho \mathbf{v} \phi) = \nabla \cdot (\Gamma \nabla \phi) + Q$$

**Transmit** term | **Convection** term | **Diffusion** term | **Source** term

Navier-Stokes equations are a special case of the general scalar equation with $\Phi = 1, u$ or $v$ and $\Gamma=0$ or $\mu$ and appropriate $Q$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \nabla \cdot \mathbf{\tau} - \nabla p + \mathbf{B}$$

$$\mathbf{\tau} = -\mu (\nabla \mathbf{v} + \nabla \mathbf{v}^T) + \frac{2}{3} \mu \mathbf{I} \nabla \cdot \mathbf{v}$$

NS equations are non-linear
- Not by itself a problem
- Can do Picard iteration

Momentum Equations $(x,y,z)$
- not a problem
- Can solve each sequentially

Source terms?
- stress tensor sources can be computed

Main Issue is that pressure is not known
- should find a way to compute it
- what PDE to use?
- Special case of incompressible flow
Options

• Option 1
  ▸ Vorticity-Stream function formulation

• Option 2
  ▸ Density Based formulation

• Option 3
  ▸ Use the Momentum equations to compute the velocity field
  ▸ Use the Continuity equation to form a pressure equation to compute the Pressure field
  ▸ This is known as the Primitive Variables Formulation
Option I

- Used in 70's-80's
  - Derive one PDE for stream function $\Psi$ and one PDE for the single component vorticity in 2D
  - Eliminates pressure as variable
  - Only 2 PDE;s as opposed to 3 for primitive variable formulation $(u,v,p)$

- However no stream function definition in 3D flows

- Vorticity-vector potential formulation in 3D
  - 6 components (3 for vorticity, 3 vector potential)
  - Fewer components for primitive variable formulation

- Difficult to derive bc's
Option 2: Density Methods

- Popular in compressible flow community
- \((u,v,w,\rho,E)\) used as unknowns
- Artificial compressibility/preconditioning schemes
- Effectively add some artificial compressibility to enable use of density-based schemes for incompressible flows
- Widely used in aeronautics industry
Option 3: Pressure Methods

• Will focus on pressure-based methods since they are the most widely used in the incompressible flow community.

• At convergence, the solution satisfies the discrete continuity and momentum equations.

• Choice of pressure-based/density-based solution technique only governs
  ▸ whether we get a solution
  ▸ how fast/cheaply we can get it
  ▸ how much memory/storage we need
SIMPLE Algorithm

- Semi-Implicit Method for Pressure-Linked Equations
- Proposed by Patankar and Spalding (1972)
- Idea is to start with discrete continuity equation
- Substitute into it the discrete u and v momentum equations
- Discrete momentum equations contain pressure differences
- Hence get an equation for the discrete pressures
- SIMPLE actually solves for a related quantity called the pressure correction
Flow in Pipes

A portion of a water-supply system is shown in the figure below. The flow rate in a pipe is given by

\[ \dot{m} = C \Delta P \]

Hydraulic conductance

Pressure drop over the length of the pipe

1. Guess values for the unknowns
2. Compute the flow rates using the guessed pressure
3. Construct a pressure correction equation, and solve for the pressure correction

\[ C_A = 0.4 \]
\[ C_B = C_D = C_F = 0.2 \]
\[ C_C = C_E = 0.1 \]
Solution

1. \( P_3 = 200 \quad P_5 = 30 \)

2. \( \dot{m}_A^* = C_A (P_1 - P_3^*) \quad \dot{m}_B^* = C_B (P_3^* - P_2) \quad \dot{m}_C^* = C_C (P_4 - P_3^*) \quad \dot{m}_D^* = C_D (P_3^* - P_5^*) \quad \dot{m}_E^* = C_E (P_6 - P_5^*) \)

3. \( \dot{m}^* = C \Delta P^* \rightarrow \sum_{k} \dot{m}_k^* \neq 0 \)
\( \dot{m} = C \Delta P \rightarrow \sum_{k} \dot{m}_k = 0 \)
\( \dot{m}' = C \Delta (P') \)

\[ \sum_{k} (\dot{m}_k^* + \dot{m}_k') = 0 \]
\( \dot{m}_A' = C_A (-p_3') \)
\( \dot{m}_B' = C_B (p_3') \)
\( \dot{m}_C' = C_C (-p_5') \)
\( \dot{m}_D' = C_D (p_3' - p_5') \)
\( \dot{m}_E' = C_E (-p_5') \)

\[ \sum_{k} \dot{m}_k' = -\sum_{k} \dot{m}_k^* \]
\[
\sum_{k=nb(3)} \left( \dot{m}_k^* + \dot{m}_k' \right) = 0
\]

\[
\dot{m}_A' - \dot{m}_B' + \dot{m}_C' - \dot{m}_D' = -\left( \dot{m}_A^* - \dot{m}_B^* + \dot{m}_C^* - \dot{m}_D^* \right)
\]

\[
C_A(-p_3') - C_B(p_3') + C_C(-p_3') - C_D(p_3' - p_5') = -\left( \dot{m}_A^* - \dot{m}_B^* + \dot{m}_C^* - \dot{m}_D^* \right)
\]

\[
\dot{m}_D' + \dot{m}_E' = -\left( \dot{m}_D^* + \dot{m}_E^* - \dot{m}_F' \right)
\]

\[
C_D(p_3' - p_5') + C_E(-p_5') = -\left( \dot{m}_D^* + \dot{m}_E^* - \dot{m}_F' \right)
\]

\[
\sum_{k=nb(5)} \left( \dot{m}_k^* + \dot{m}_k' \right) = 0
\]
\[-0.7(p'_3) + 0.2(p'_5) = 10\]
\[0.2(p'_3) - 0.3(p'_5) = -15\]  
\[\Rightarrow \begin{cases} p'_3 = 0 \\ p'_5 = 50 \end{cases}\]

\[\begin{array}{ll}
\dot{m}_A = 0 & \dot{m}_B = -14 \\
\dot{m}'_A = 0 & \dot{m}'_B = 0 \\
\dot{m}_C = -20 & \dot{m}'_C = 0 \\
\dot{m}_D = 34 & \dot{m}'_D = -10 \\
\dot{m}_E = 1 & \dot{m}'_E = -5 \\
\end{array}\]
SIMPLE Based Algorithms
The SIMPLE Algorithm

\[ \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ \nabla \cdot (\rho \mathbf{v}) = -\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{B} \]

\[ a_c \mathbf{v}_c + \sum_{F=NB(C)} a_F \mathbf{v}_F = -V_c \left( \nabla p^{(n)} \right)_C \]

\[ \ldots \mathbf{v}^*, P^{(n)} \]

Predictor

\[ \sum_{f=nb(C)} m_f^e \neq 0 \quad \sum_{f=nb(C)} m_f = 0 \]

\[ \sum_{f=nb(C)} \left( m_f^e + m_f' \right) = 0 \quad m_f^e = \rho \mathbf{v}_f^* \cdot \mathbf{S}_f \quad m_f' = \rho \mathbf{v}_f' \cdot \mathbf{S}_f \]

\[ \sum_{f=nb(C)} m_f' = -\sum_{f=nb(C)} m_f^e \]

Corrector

\[ a_e^u u_e^* + \sum_{f=NB(e)} a_f^u u_f^* = -V_e \left( \nabla p^{(n)} \right)_e \]

\[ \sum_{f=NB(C)} \rho \mathbf{v}_f^* \cdot \mathbf{S}_f = \rho u_e^* \mathbf{A}_e - \rho u_w^* \mathbf{A}_w \]

\[ u_e^* + \sum_{f=NB(e)} \frac{a_f^u}{a_e^u} u_f^* = -\frac{V_e}{a_e^u} \left( \frac{\partial p^{(n)}}{\partial x} \right)_e \]

\[ u_w^e = H_{w} \left( u_e^* \right) - d_e \left( \frac{\partial p^{(n)}}{\partial x} \right)_{w} \]

\[ u_e^* = H_{e} \left( u_e^* \right) - d_e \left( \frac{\partial p^{(n)}}{\partial x} \right)_{e} \]

\[ u_e' = H_{f} \left( u_e' \right) - d_f \left( \nabla p \right)_{f} \]

\[ \mathbf{v}_f' = H_{f} \left( \mathbf{v}' \right) - d_f \left( \nabla p \right)_{f} \]
Pressure Equation

\[ \sum_{f=nb(C)} \rho v'_f \cdot S_f = - \sum_{f=nb(C)} \dot{m}_f \]

Substituting \( H_f (v') - d_f (\nabla p')_f \)

\[ \sum_{f=nb(C)} \left[ H_f (v') - d_f (\nabla p')_f \right] \cdot S_f = - \sum_{f=nb(C)} \dot{m}_f \]

\[ \sum_{f=nb(C)} -d_f (\nabla p')_f \cdot S_f = - \sum_{f=nb(C)} \dot{m}_f - \sum_{f=nb(C)} \left\{ \sum_{f=nb(C)} \left[ -d_f (\nabla p')_f \cdot S_f \right] \right\} \]

\[ \sum_{f=nb(C)} -d_f (\nabla p')_f \cdot S_f = \sum_{f=nb(C)} -d_f A_f \left( \frac{p'_f - p'_c}{\|CF\|} \right) - \sum_{f=nb(C)} \dot{m}_f = -\left( \rho u_e A_e - \rho u_w A_w \right) \]

\[ \Rightarrow \sum_{f=nb(C)} -d_f A_f \left( \frac{p'_f - p'_c}{\|CF\|} \right) = -\left( \rho u_e A_e - \rho u_w A_w \right) \]

\[ a_c^p p'_c + (a_e^p p'_e + a_w^p p'_w) = b_C^p \quad \Rightarrow v'_f = -d_f (\nabla p')_f \quad \Rightarrow v''_f = v'_f - d_f (\nabla p')_f \]
The SIMPLE Algorithm

\[ a_C^r v_C^* + \sum_{F=NB(C)} a_F^r v_F^* = -V_C (\nabla p_C^{(n)}) \]
\[ \therefore v^*, P^{(n)} \]

\[ a_C^p p_C^* + (a_E^p p_E^* + a_W^p p_W^*) = b_C^p \]
\[ \therefore v^{**}, P' \]
\[ p_C^* = p_C^{(n)} + p_C' \]
\[ v^{**}_f = v'^*_f - d_f (\nabla p')_f \]

start with \( n \)th iteration values \( \dot{m}_f^{(n)}, v^{(n)}, \rho^{(n)}, P^{(n)} \)

\[ \text{assemble and solve} \] momentum equation for \( v^* \)

\[ \text{assemble} \] \( \dot{m}_f^* \) using Rhie-Chow interpolation

\[ \text{assemble and solve} \] Pressure correction equation for \( P' \)

\[ \text{correct} \] \( \dot{m}_f^*, v^*, \rho^{(n)} \) and \( P^{(n)} \) to get improved \( \dot{m}_f^{**}, v^{**}, \rho^* \) and \( P^* \)

repeat until convergence
Sequential Solution

- Pressure methods were also developed as sequential methods as opposed to coupled methods where all the Navier-Stokes equation are solved simultaneously.

- In sequential methods each of the following equation is solved individually in sequence:
  - Solve u-momentum equation for u velocity component, with all other variables assumed known
  - Solve v-momentum equation for v velocity component
  - Modify the continuity equation to yield a pressure equation and solve for pressure
  - Repeat process until convergence
Collocated Grids
\[ \sum_{f = nb(C)} \dot{m}_f' = - \sum_{f = nb(C)} \dot{m}_f^* \]

\[ \sum_{f = nb(C)} \rho \vec{v}_f' \cdot \vec{S}_f = - \sum_{f = nb(C)} \dot{m}_f^* \]

\[ \vec{v}_f' = \frac{1}{2} (\vec{v}' + \vec{v}'_E) \]

\[ \frac{1}{2} \left( H_C(\vec{v}') - D_C \left( \frac{\partial p'}{\partial x} \right)_C + H_F(\vec{v}') - D_F \left( \frac{\partial p'}{\partial x} \right)_F \right) \]

\[ \bar{u}_e = \bar{H}_e(u') + D_e \left( \frac{\partial p'}{\partial x} \right)_e \]

\[ \bar{u}_w = \bar{H}_w(u') + D_w \left( \frac{\partial p'}{\partial x} \right)_w \]

\[ \bar{u}_e - \bar{u}_w = \bar{H}_e(u') - \bar{H}_w(u') + D_e \left( \frac{\partial p'}{\partial x} \right)_e - D_w \left( \frac{\partial p'}{\partial x} \right)_w \]

\[ \approx D \frac{1}{2} \left[ \left( \frac{\partial p'}{\partial x} \right)_E + \left( \frac{\partial p'}{\partial x} \right)_C - \left( \frac{\partial p'}{\partial x} \right)_C - \left( \frac{\partial p'}{\partial x} \right)_W \right] \]

\[ \approx \frac{D}{\Delta x} \frac{1}{2} \left[ p'_{EE} - 2p'_{C} + p'_{WW} \right] \]

Similarly,

\[ \dot{m}_e - \dot{m}_w = \rho A (u_E - u_W) \]

\[ a_C p'_C + \alpha_{EE} p'_{EE} + \alpha_{WW} p'_{WW} = - (\dot{m}_E - \dot{m}_W) \]
Rhie Chow Interpolation

\[ \mathbf{v}_P = H_C (\mathbf{v}) - D_C \nabla p_C \]
\[ \mathbf{v}_F = H_F (\mathbf{v}) - D_F \nabla p_F \]
\[ \mathbf{v}_f = \bar{H}_f (\mathbf{v}) - \bar{D}_f \nabla p_f \]

1. \[ \bar{H}_f (\mathbf{v}) = g_P H_P (\mathbf{v}) + g_F H_F (\mathbf{v}) \]
   \[ = g_P (\mathbf{v}_P + D_P \nabla p_P) + g_F (\mathbf{v}_F + D_F \nabla p_F) \]
   \[ \approx \bar{\mathbf{v}}_f + \bar{D}_f \nabla p_f \]

2. \[ \mathbf{v}_f = (\bar{\mathbf{v}}_f + \bar{\nabla} p_f) - \bar{D}_f \nabla p_f \]
   \[ = \bar{\mathbf{v}}_f - \bar{D}_f (\nabla p_f - \bar{\nabla} p_f) \]

3. \[ \dot{m}_f = \rho \mathbf{v}_f^* \cdot S_f \]
   \[ \dot{\mathbf{m}}_f = \rho \mathbf{v}_f' \cdot S_f \]
Pressure Correction

\[
\sum_{f=nb(C)} \dot{m}'_f = - \sum_{f=nb(C)} \dot{m}^*_f
\]

\[
\sum_{f=nb(C)} \rho v'_f \cdot S_f = - \sum_{f=nb(C)} \dot{m}^*_f
\]

\[
v'_f = \bar{v}'_f - \bar{D}_f \left( \nabla p'_f - \nabla \bar{p}_f \right)
\]

\[
\begin{align*}
\dot{u}'_e &= \bar{u}'_e - \bar{D}_e \left( \left( \frac{\partial p'}{\partial x} \right)_e - \left( \frac{\partial p'}{\partial x} \right)_c \right) \\
\dot{u}'_w &= \bar{u}'_w - \bar{D}_w \left( \left( \frac{\partial p'}{\partial x} \right)_w - \left( \frac{\partial p'}{\partial x} \right)_c \right)
\end{align*}
\]

\[
\begin{align*}
\dot{u}'_e - \dot{u}'_w &= \left[ \bar{u}'_e - \bar{D}_e \left( \left( \frac{\partial p'}{\partial x} \right)_e - \left( \frac{\partial p'}{\partial x} \right)_c \right) \right] - \left[ \bar{u}'_w - \bar{D}_w \left( \left( \frac{\partial p'}{\partial x} \right)_w - \left( \frac{\partial p'}{\partial x} \right)_c \right) \right] \\
&\approx \bar{D} \frac{1}{2} \left[ \left( \frac{\partial p'}{\partial x} \right)_e - \left( \frac{\partial p'}{\partial x} \right)_w \right] \\
&\approx \frac{D}{\Delta x} \frac{1}{2} \left[ p'_e - 2 p'_c + p'_w \right]
\end{align*}
\]

Similarly,

\[
\dot{m}^*_e - \dot{m}^*_w = \rho_j u^*_e A_e - \rho_w u^*_w A_w
\]

\[
a^p_{c} p'_c + a^p_{E} p'_E + a^p_{w} p'_w = - \sum_{f=nb(C)} \dot{m}^*_f
\]
The Pressure-Correction Equation

\[ a_c^* v_c^* + \sum_{F=NB(C)} a_F^* v_F^* = -V_C \left( \nabla p^{(n)}_C \right) \]

Assemble and solve momentum equation for \( v^* \)

\[ \dot{m}_f^* = \rho_f^{(n)} v_f^* \cdot S_f = \rho_f^{(n)} \left[ v_f^* - \bar{D}_f \left( \nabla p'_f - \nabla p'_f \right) \right] \cdot S_f \]

\[ \dot{m}_f' = -\rho_f^{(n)} \left( \bar{D}_f \nabla p'_f \right) \cdot S_f \]

\[ v_f' = -\bar{D}_f \nabla p'_f \]

Assemble and solve Pressure Correction equation for \( P' \)

\[ a_C^p p'_C + (a_E^p p'_E + a_W^p p'_W) = b_C^p \]

\[ \therefore v^{**}, P' \]

\[ p_c^* = p_c^{(n)} + p'_C \]

\[ v_f^{**} = v_f' - \bar{d}_f \left( \nabla p'_f \right) \]

Assemble \( \dot{m}_f^* \) using Rhie-Chow interpolation

Correct \( \dot{m}_f^*, v^* \), \( \rho^{(n)} \) and \( P^{(n)} \) to get improved \( m_f^*, v^{**}, \rho^* \) and \( P^* \)

Repeat until convergence

\[ \therefore v^{(n)}, P^{(n)} \]
Pressure-Velocity Coupling

collocated variable arrangement
Rhie-Chow Interpolation

First Predictor

\( v_p' = H[v'] J_p - D_p' (\nabla P') \)

First Corrector

\( (v', P', \rho') (v'' = v' + v', P' = P^{(0)} + P', \rho' = \rho^{(0)} + \rho') \)

\( v_p'' = H[v''], J_p = H[v' + v'] - D_p' [\nabla (P^{(0)} + P')] \)

\( [\rho' = C \rho P'] \)

Second Predictor

\( v_p'' = H''[v''], J_p = [\nabla (P'' + P')] \)

Second Corrector

\( (v'', P'', \rho'') \)

\( v_p''' = H'''[v''' = v'' + P'' - P' + \rho'' + \rho''] \)

Condition

\[ \frac{\rho'' - \rho'}{\delta t} V + \Delta (\rho' v'' S) = 0 \]

\[ \frac{\rho'' - \rho'}{\delta t} V + \Delta [\rho'\nabla v'' - (\rho'' - \rho') V - \Delta (\rho' v'') S] = 0 \]

\[ \frac{V C_p}{\delta t} P' + \Delta [C_p U'' P'] - \Delta [\rho'' \nabla (v'') S'] - \Delta (\rho'' v'') S = 0 \]

Approximation

Neglect \( H[v] \)
\[ \Delta [(\rho' v) S] \quad \Rightarrow \quad v_p' = -D_p' (\nabla P') \]

Approximation Equation

\[ \frac{V C_p}{\delta t} P' + \Delta [C_p U'' P'] - \Delta [\rho'' \nabla (v'') S'] - \Delta (\rho'' v'') S = 0 \]

SIMPLE

SIMPLEC

SIMPLER

SIMPLEST

SIMPLE-M

PISO

treatment leads to variety of schemes
Problem 1 - Staggered Grid

• Use the SIMPLE procedure to compute $p_2$, $u_B$, and $u_C$ from the following data:

$$\Delta x = 2, \ c_B = 0.25, c_C = 0.2$$
$$A_B = 5, A_C = 4, p_1 = 200, p_3 = 38$$

• As an initial guess, set

$$u_B = u_C = 15 \text{ and } p_2 = 120$$